The Road Closure Problem

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Abstract

About 15 years ago, João Araújo first interested me in problems of the following form, linking permutation groups with transformation semigroups:

> Let P be a property of semigroups, such as *regularity*, or a property of transformation semigroups, such as *synchronization*. Which finite permutation groups G on a set Ω have the property that, for any map $f : \Omega \to \Omega$ which is not a permutation, the semigroup $\langle G, f\rangle$ has property P? What if we weaken the assumption to require this just for all maps f of some given rank k?

In this way we have uncovered some remarkable properties of permutation groups, often lying between primitivity and 2-transitivity, which are worth studying, and have obtained many results about these.

One case where we have not been able to obtain a complete classification of the permutation groups, despite a lot of effort, is: Which finite permutation groups G on a set Ω have the property that, for any map f of rank 2, the semigroup $\langle G, f\rangle \setminus G$ is idempotent-generated? We showed that this is equivalent to the *Road Closure Property* of G, which asserts the following:

> For any orbit $\mathcal{O}$ for G on the set of 2-element subsets of Ω, and any proper block of imprimitivity B for G on the set of edges of Γ, the graph with edge set $\mathcal{O} \setminus B$ is connected.

It is known that a group with this property is primitive, and basic (that is, preserves no Cartesian product structure on Ω, and so is affine, diagonal or almost simple. We conjecture that the affine and

diagonal groups do not occur, and that in the almost simple case the stabiliser of B has index 2 or 3 in G. There are a number of interesting examples, including some derived from triality.

I will report on progress on this conjecture.