NONCOPRIME FIXED POINT FREE ACTION OF A NILPOTENT GROUP

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An active area of research

Throughout all groups are finite.

Question

Let A act on G by automorphisms. How does the nature of the action of A influence the structure of both A and G?

Fixed point subgroup

Definition

An element $g \in G$ is called a fixed point of A in G if $g^a = g$ holds for all $a \in A$.

$C_G(A) = \{g \in G : g^a = g \text{ holds for all } a \in A\} = \bigcap_{a \in A} C_G(a)$ is the set of all fixed points of A in G called the fixed point subgroup.

We call the action FIXED POINT FREE whenever $C_G(A) = 1$.

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An active area of research-Length type problems

Problems of finding some bounds for the invariants of a solvable group, like the derived length, p-length, Fitting length by using the known information about the group.

(started by Hall-Higman in 1956)

We essentially focus on some conjectures belonging to length type problems in the theory of finite groups.

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Define

$$F_0(G) := \langle 1 \rangle$$
$$F_1(G) := F(G)$$
$$F_i(G)/F_{i-1}(G) := F\left(G/F_{i-1}(G)\right) \quad \text{for } i > 1.$$

For a nontrivial solvable G, $F(G)$ is also nontrivial and therefore, there exists a smallest positive integer n such that $G = F_n(G)$.

This n is called the Fitting (or nilpotent) length of G and is denoted by $h(G)$.

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Definition

Any nilpotent self-normalizing subgroup of a solvable group is called a Carter subgroup.

Let $\ell(C)$ denote the number of primes, counted with multiplicities, dividing $|C|$.

(Thompson's conjecture) Let C be a Carter subgroup of the solvable group H. Then $h(H) \leq f(\ell(C))$ for some function f.

Dade established this conjecture in 1969 by proving that

Theorem

Let C be a Carter subgroup of the solvable group H. Then

$$h(H) \leq 10(2^{\ell(C)} - 1) - 4\ell(C).$$

* E. C. Dade, *Carter subgroups and Fitting heights of finite solvable groups*, Illinois J. Math 13 (1969) 449-513.

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Theorem

Let H be a solvable group whose Carter subgroups have a normal complement H. If C is a Carter subgoup of H, then

$$h(H) \leq 5(2^{\ell(C)} - 1).$$

Remark

Let A be a nilpotent group acting on the solvable group G fixed point freely. Then

A is a Carter subgroup of the semidirect product $H = G \rtimes A$ with normal complement G, because $[A, N_G(A)] \leq A \cap N_G(A) = 1$ and so $N_H(A) = C_G(A)A = A$.

Furthermore, A acts fixed point freely on any A-invariant section U of G since A is also Carter subgroup of the semidirect product UA.

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Hence as a corollary of Dade's theorem above we have

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Expectations

Let H be a solvable group whose Carter subgroups have a normal complement G. If C is a Carter subgoup of H, then there is a linear function f such that

$$h(H) \leq f(\ell(C)).$$

Correspondingly,

Let A be a nilpotent group acting fixed point freely on the solvable group G. Then there is a linear function f such that

$$h(G) \leq f(\ell(A)).$$

Coprime and noncoprime actions

Let G be a group and A a group acting on G by automorphisms. We call the action of A on G

- *a coprime action when $(|G|, |A|) = 1$*
- *a noncoprime action when $(|G|, |A|) \neq 1$ is allowed.*

Under coprime action, there are some useful relations between G and A, which are not valid in general. That is why the coprime action is mostly studied.

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Consequences of coprime action

When the action is coprime , the following hold and make some inductive arguments quite easy to apply:

- $C_{G/N}(A) = C_G(A)N/N$ *for every* A*-invariant normal subgroup* N *of* G.
- $G = [G, A]C_G(A)$

(Here $[G, A] = \langle g^{-1}g^a \; : \; g \in G, a \in A \rangle$ *and*

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Glauberman's lemma

Let A act coprimely on the group G. Suppose that A and G each act on a nonempty set Ω. Assume further that the semidirect product $G \rtimes A$ acts on Ω and that G is transitive on Ω. Then
(a) there exists an A-invariant element $w \in \Omega$,
(b) if $u, w \in \Omega$ are A-invariant, then there is $c \in C_G(A)$ such that $u.c = w$.

Schur-Zassenhauss theorem: Let $N \lhd G$ be such that $(|N|, |G/N|) = 1$. Then there exists $L \leq G$ such that $G = NL$ and $N \cap L = 1$.

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Some best possible bounds on nilpotent lengths under coprime action

Theorem

Let A and G be both solvable. If A acts coprimely on G then

$$h(G) \leq h(C_G(A)) + 2\ell(A)$$

and this bound is the best possible.

* A. Turull, *Fitting height of groups and of fixed points,* J. Algebra 86 (1984) 555-566.

Some best possible bounds on Fitting lengths under coprime action

Theorem

(Turull, 1990) Let A act coprimely on the solvable group G. Suppose that every proper subgroup of A acts with regular orbits, that is, for every proper subgroup B of A and every B-invariant section S of G such that B acts irreducibly on S there exists $v \in S$ such that $C_B(v) = C_B(S)$. Then

$$h(G) \leq \ell(C_G(A)) + \ell(A)$$

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* A. Turull, *Groups of automorphisms and centralizers.* Math. Proc. Cambridge Philos. Soc. 107 (1990) 227-238.

Fixed point free coprime action

Theorem

Let A act fixed point freely on G where $|A| = 2$. Then G is an abelian group of odd order.

Theorem

Let A act fixed point freely on G where $|A| = 3$. Then G is a nilpotent group of class ≤ 2.

Frobenius' Conjecture

Let A act on G fixed point freely. If $|A|$ is a prime, then G is solvable and nilpotent, that is, $h(G) = 1$.
(proven by Thompson in 1959).

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Let A be a group acting fixed point freely and coprimely on the group G. Then G is solvable.

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Remark

- *By Turull's first result, if A is solvable and A acts on G coprimely and fixed point freely then $h(G) \leq 2\ell(A)$.*
- *For each A, there is a solvable G such that A acts coprimely and fixed point freely on G and $h(G) = \ell(A)$. Thus the second result of Turull gives the best possible bound under the above assumptions.*

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RECALL: Dade proved the following.

Theorem

Let a nilpotent group A act fixed point freely on the group G. Then

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and he conjectured that there is a linear function f such that $h(G) \leq f(\ell(A))$. No linear bound is known.

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Some results improving the bound of Dade in special cases

Turull, 1995

Let an abelian group A of squarefree exponent act fixed point freely on the group G. Then $h(G) \leq 5\ell(A)$

Ercan, Güloğlu -2008, 2009

Let a group A act on the group G fixed point freely. If $|A|$ is coprime to 6 and the Sylow 2-subgroups of G are abelian, then

- $h(G) \leq 2^{\ell(A)} - 1$, when A is nilpotent
- $h(G) \leq 2^{\ell(A)} - \ell(A) - 1$, when A is abelian .

Jabara-2018

Let G be a group and let A be cyclic acting fixed point freely on the solvable group G. Then $h(G) \leq 7\ell(A)^2$.

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* A. Turull, *Character theory and length problems*, Finite and locally finite groups (Istanbul 1994), 377-400, Nato Adv.Sci.Inst.Ser.C 471 Kluwer Acad.Publ. 1995.

* G. Ercan and İ. Ş. Güloğlu, *Fixed point free action on groups of odd order,* J. Algebra 320 (1) (2008) 426-436.

* E. Jabara, *The Fitting length of finite soluble groups II: Fixed point free automorphisms,* J. Algebra 487 (2017) 161-172.

More on fixed point free noncoprime action

Theorem

Given nonnilpotent A and a positive integer k. Then there is a solvable G such that A acts fixed point freely and noncoprimely on G, and $h(G) \geq k$.

* S. D. Bell and B. Hartley, *A note on fixed point free actions of finite groups,* Quart. T. Math. Oxford (2) 41 (1990) 127-130.

Conjecture

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Some partial answers

In 1987, C. Kei-Nah established this conjecture when A is cyclic of order pq for primes p, q.

Ercan-Güloğlu (2004-2010) Conjecture is true when

- A is cyclic of order pqr for pairwise distinct primes p, q, r
- A is abelian of square free exponent coprime to 6 and G is of odd order.
- A is abelian of square free odd exponent coprime to all Fermat primes or
  A is abelian of square free exponent coprime to 6 and the Sylow 2-subgroups of G are abelian.
- A is abelian of order pqr for odd primes p, q, r which are not necessarily distinct and the Sylow 2-subgroups of G are abelian
- A is cyclic of order $p^n q$ for primes p, q coprime to 6 and the Sylow 2-subgroups of G are abelian

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- A is abelian of order pqr for odd primes p, q, r which are not necessarily distinct and the Sylow 2-subgroups of G are abelian
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Some partial answers

In 1987, C. Kei-Nah established this conjecture when A is cyclic of order pq for primes p, q.

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Ercan-Güloğlu - 2023

- Let A be a nilpotent group acting fixed point freely on the group G where $(|F_3(G)/F(G)|, |A|) = 1$ and $|F_2(G)/F(G)|$ is odd. Suppose further that A is $\mathbb{Z}_2 \wr \mathbb{Z}_2$-free and $\mathbb{Z}_r \wr \mathbb{Z}_r$-free for all Mersenne primes r. Then $h(G) \leq \ell(A)$.
- Let A be a cyclic group acting fixed point freely on the (solvable) group G. Then

  (i) $h(G) \leq \ell(A)$ if $(|F_3(G)/F_2(G)|, |A|) = 1$ and $|F_2(G)/F(G)|$ is odd, and

  (ii) $h(G) \leq 2\ell(A)$ if $|F_2(G)/F(G)|$ is odd.

* A.Espuelas

Let A be a nilpotent group acting fixed point freely on G. Then there exist sections $P_1, \ldots, P_h$ of G (called an A-Fitting chain of G) with $P_i = S_i/T_i$, $i = 1, \ldots, h$, where $S_i, T_i \leq G$ such that $T_i \lhd S_i$ and $h = h(G)$ satisfying

(a) P_i *is a nontrivial* p_i*-group, for some prime* p_i,

(b) $\Phi(P_i) \leq Z(P_i), \Phi(\Phi(P_i)) = 1$ *and if* $p_i \neq 2$, $exp(P_i) = p_i$,

(c) P_i *is* A*-invariant, for* $i = 1, \ldots, h$

(d) $p_i \neq p_{i+1}$, *for* $i = 1, \ldots, h-1$,

(e) $T_i =$ *Ker*$(S_i$ *on* $P_{i+1})$, *for* $i = 1, \ldots, h-1$,

(f) $T_h = 1$ *and* $S_h \leq F(G)$,

(g) $[\Phi(P_{i+1}), S_i] = 1$, *for* $i = 1, \ldots, h-1$,

(h) $(\prod_{1 \leq j < i} S_j)A$ *acts irreducibly on* $\tilde{P}_i$.

Theorem

Let $G \trianglelefteq GA$ where G is solvable, A satisfies the condition $**$ and normalizes a Sylow system of G.

Let P be a p-subgroup of G, for some prime p, such that $P \trianglelefteq GA$, $\Phi(P) = P' \leq Z(P)$, $exp(P) = p$ if $p \neq 2$, and $P/\Phi(P)$ is completely reducible GB-module for any subgroup B of A.

Let Q be an A-invariant q-subgroup of $C_G(\Phi(P)$ for a prime q which is coprime to $p|A|$and not a Fermat prime if $p = 2$. Assume that $\Phi(Q/Q_0) \leq Z(Q/Q_0)$ where $Q_0 = C_Q(P)$, and $QC_G(P/P') \trianglelefteq GA$, and that $[Q, P] = P$ if $P' \neq 1$.

Assume further that the following hold:

(i) $P = [P, B]^G$ for every $B \leq A$ with $\ell(B) \geq 1$,

(ii) if P is nonabelian, $Q = [Q, C]^{N_G(Q)} Q_0$ for every $C \leq A$ with $\ell(C) \geq 2$,

(iii) p is coprime to $|A|$ when A is noncyclic.

Let χ be a complex GA-character such that $P \not\leq \ker(\chi)$. Then

χ_A contains the regular A-character.

A new concept: GOOD ACTION

Definition

Let A act on G by automorphisms. We say the action is good, if

$$H = [H, B]C_H(B)$$

for every subgroup B of A and every B-invariant subgroup H of G.

Every coprime action is good.

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Every coprime action is good.

Consequences of good action

Let the action of A on G be good. If $B \leq A$ and N is a normal B-invariant subgroup of G then

- *The action of B on every B-invariant subgroup of G is good.*
- *The equality $[H, B, B] = [H, B]$ holds for any B-invariant subgroup H of G.*
- *The equality $C_{H/N}(B) = C_H(B)N/N$ holds for any B-invariant subgroup H of G containing N.*
- *The induced action of B on G/N is good.*
- *Let $p \in \pi(A)$ and let B be a p-subgroup of A. If G is p-solvable then $[G, B]$ is a p'-group. In particular, B acts trivially on each B-invariant p-subgroup of G.*
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Example of a noncoprime nongood action

Nontrivial action of a p-group on another p-group is not good.

For example, let $G = Q_8 = \langle a, b : a^4 = 1, a^2 = b^2, a^b = a^{-1} \rangle$ and let $\alpha \in Aut\ Q_8$ defined by $\alpha(a) = b$ and $\alpha(b) = a$.

Set $A = \langle \alpha \rangle$. Then $N = \langle a^2 \rangle$ is an A-invariant normal subgroup of G. We have $C_{G/N}(A) \neq 1$, but $C_G(A) = N$. This means

$$C_{G/N}(A) \neq C_G(A)N/N$$

and hence the action is not good.

Some extensions of coprime results to good action

Theorem

Let A and G be both solvable where G is of odd order and let A act on G. If the action is good then

$$h(G) \leq h(C_G(A)) + 4\ell(A).$$

* G. Ercan, İ. Ş. Güloğlu and E. Jabara, *Good action on a finite group*,
J. Algebra 560 (2020) 486-501.

Some extensions of coprime results to good action

Theorem

Let a nilpotent group A of odd order which is $C_p \wr C_p$-free for any prime p act good on the group G of odd order. Assume further that every proper subgroup of A acts with regular orbits. Then

$$h(G) \leq \ell(A) + \ell(C_G(A))$$

and this bound is the best possible.

* G. Ercan and İ. Ş. Güloğlu, *Good action of a nilpotent group with regular orbits*, Com. Algebra 50 (10) (2022) 4191-4194.

Fixed point free good action

Theorem

Let A be a nilpotent group of odd order acting good and fixed point freely on the group G. Suppose that A is $C_q \wr C_q$-free for any prime q, and that every subgroup of A acts with regular orbits on G. Then

$$h(G) \leq \ell(A).$$

* G. Ercan, İ. Ş. Güloğlu and E. Jabara, *Good action on a finite group*, J. Algebra 560 (2020) 486-501.

THANK YOU