Diagonal structures and beyond

R. A. Bailey
University of St Andrews
QMUL (emerita)

Ural Seminar on Group Theory and Combinatorics,
25 January 2022

Outline

1. Partitions

Outline

1. Partitions
2. Some statistical history

Outline

1. Partitions
2. Some statistical history
3. Diagonal semilattices

Outline

1. Partitions
2. Some statistical history
3. Diagonal semilattices
4. Diagonal graphs

Outline

1. Partitions
2. Some statistical history
3. Diagonal semilattices
4. Diagonal graphs
5. … and beyond.

Chapter 1

Partitions

What is a Latin square?

Definition

Let n be a positive integer.
A Latin square of order n is an $n \times n$ array of cells in which n symbols are placed, one per cell, in such a way that each symbol occurs once in each row and once in each column.

What is a Latin square?

Definition

Let n be a positive integer.
A Latin square of order n is an $n \times n$ array of cells in which n symbols are placed, one per cell, in such a way that each symbol occurs once in each row and once in each column.

The symbols may be letters, numbers, colours, …

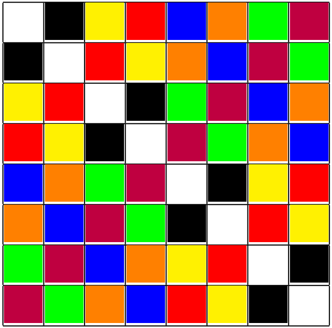
What is a Latin square?

Definition

Let n be a positive integer.
A Latin square of order n is an $n \times n$ array of cells in which n symbols are placed, one per cell, in such a way that each symbol occurs once in each row and once in each column.

The symbols may be letters, numbers, colours, …

A Latin square of order 8



Partitions

Definition

A partition of a set Ω is a set P of pairwise disjoint non-empty subsets of Ω, called parts, whose union is Ω.

Partitions

Definition

A partition of a set Ω is a set P of pairwise disjoint non-empty subsets of Ω, called parts, whose union is Ω.

Definition

A partition P is uniform if all of its parts have the same size, in the sense that, whenever Γ_1 and Γ_2 are parts of P, there is a bijection from Γ_1 onto Γ_2.

Partitions

Definition

A partition of a set Ω is a set P of pairwise disjoint non-empty subsets of Ω, called parts, whose union is Ω.

Definition

A partition P is uniform if all of its parts have the same size, in the sense that, whenever Γ_1 and Γ_2 are parts of P, there is a bijection from Γ_1 onto Γ_2.

Example

If Ω is the set of cells in a Latin square, then there are five natural uniform partitions of Ω:

R	each part is a row;
C	each part is a column;
L	each part consists of the those cells with a given letter;
U	the universal partition, with a single part;
E	the equality partition, whose parts are singletons.

The partial order on partitions of a set

A natural partial order on partitions of a set is defined by

$P \preccurlyeq Q$ if and only if every part of P is contained in a part of Q.

The partial order on partitions of a set

A natural partial order on partitions of a set is defined by

$P \preccurlyeq Q$ if and only if every part of P is contained in a part of Q.

So $E \preccurlyeq P \preccurlyeq U$ for all partitions P.

The partial order on partitions of a set

A natural partial order on partitions of a set is defined by

$P \preccurlyeq Q$ if and only if every part of P is contained in a part of Q.

So $E \preccurlyeq P \preccurlyeq U$ for all partitions P.

Definition

The infimum, or meet, of partitions P and Q
is the partition $P \wedge Q$ each of whose parts is
a non-empty intersection of a part of P and a part of Q.

The partial order on partitions of a set

A natural partial order on partitions of a set is defined by

$P \preccurlyeq Q$ if and only if every part of P is contained in a part of Q.

So $E \preccurlyeq P \preccurlyeq U$ for all partitions P.

Definition

The infimum, or meet, of partitions P and Q
is the partition $P \wedge Q$ each of whose parts is
a non-empty intersection of a part of P and a part of Q.
So $P \wedge Q \preccurlyeq P$ and $P \wedge Q \preccurlyeq Q$;

The partial order on partitions of a set

A natural partial order on partitions of a set is defined by

$P \preccurlyeq Q$ if and only if every part of P is contained in a part of Q.

So $E \preccurlyeq P \preccurlyeq U$ for all partitions P.

Definition

The infimum, or meet, of partitions P and Q
is the partition $P \wedge Q$ each of whose parts is
a non-empty intersection of a part of P and a part of Q.
So $P \wedge Q \preccurlyeq P$ and $P \wedge Q \preccurlyeq Q$;
and if $S \preccurlyeq P$ and $S \preccurlyeq Q$ then $S \preccurlyeq P \wedge Q$.

The partial order on partitions of a set

A natural partial order on partitions of a set is defined by

$P \preccurlyeq Q$ if and only if every part of P is contained in a part of Q.

So $E \preccurlyeq P \preccurlyeq U$ for all partitions P.

Definition

The infimum, or meet, of partitions P and Q
is the partition $P \wedge Q$ each of whose parts is
a non-empty intersection of a part of P and a part of Q.
So $P \wedge Q \preccurlyeq P$ and $P \wedge Q \preccurlyeq Q$;
and if $S \preccurlyeq P$ and $S \preccurlyeq Q$ then $S \preccurlyeq P \wedge Q$.

Definition

The supremum, or join, of partitions P and Q
is the partition $P \vee Q$ which satisfies $P \preccurlyeq P \vee Q$ and $Q \preccurlyeq P \vee Q$
and if $P \preccurlyeq S$ and $Q \preccurlyeq S$ then $P \vee Q \preccurlyeq S$.

The partial order on partitions of a set

A natural partial order on partitions of a set is defined by

$P \preccurlyeq Q$ if and only if every part of P is contained in a part of Q.

So $E \preccurlyeq P \preccurlyeq U$ for all partitions P.

Definition

The infimum, or meet, of partitions P and Q
is the partition $P \wedge Q$ each of whose parts is
a non-empty intersection of a part of P and a part of Q.
So $P \wedge Q \preccurlyeq P$ and $P \wedge Q \preccurlyeq Q$;
and if $S \preccurlyeq P$ and $S \preccurlyeq Q$ then $S \preccurlyeq P \wedge Q$.

Definition

The supremum, or join, of partitions P and Q
is the partition $P \vee Q$ which satisfies $P \preccurlyeq P \vee Q$ and $Q \preccurlyeq P \vee Q$
and if $P \preccurlyeq S$ and $Q \preccurlyeq S$ then $P \vee Q \preccurlyeq S$.
Draw a graph by putting an edge between two points if they are in the same part of P or the same part of Q. Then the parts of $P \vee Q$ are the connected components of the graph.

Hasse diagrams

Given a collection $\mathcal{P}$ of partitions of a set Ω,
we can show them on a Hasse diagram.

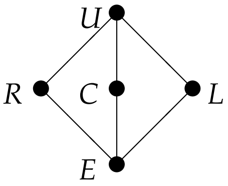
- Draw a dot for each partition in $\mathcal{P}$.
- If $P \prec Q$ then put Q higher than P in the diagram.
- If $P \prec Q$ but there is no S in $\mathcal{P}$ with $P \prec S \prec Q$ then draw a line from P to Q.

Hasse diagrams

Given a collection $\mathcal{P}$ of partitions of a set Ω,
we can show them on a Hasse diagram.

- Draw a dot for each partition in $\mathcal{P}$.
- If $P \prec Q$ then put Q higher than P in the diagram.
- If $P \prec Q$ but there is no S in $\mathcal{P}$ with $P \prec S \prec Q$ then draw a line from P to Q.

Here is the Hasse diagram for a Latin square.



An alternative definition of Latin square

Definition

Let P and Q be uniform partitions of a set Ω. Then P and Q are compatible if

- whenever ω_1 and ω_2 are points in the same part of $P \vee Q$, there are points α and β such that
  - ω_1 and α are in the same part of P,
  - α and ω_2 are in the same part of Q,
  - ω_1 and β are in the same part of Q,
  - β and ω_2 are in the same part of P.
- $P \wedge Q$ is uniform.

An alternative definition of Latin square

Definition

Let P and Q be uniform partitions of a set Ω. Then P and Q are compatible if

- whenever ω_1 and ω_2 are points in the same part of $P \vee Q$, there are points α and β such that
  - ω_1 and α are in the same part of P,
  - α and ω_2 are in the same part of Q,
  - ω_1 and β are in the same part of Q,
  - β and ω_2 are in the same part of P.
- $P \wedge Q$ is uniform.

Definition

A Latin square is a set $\{R, C, L\}$ of pairwise compatible uniform partitions of a set Ω which satisfy $R \wedge C = R \wedge L = C \wedge L = E$ and $R \vee C = R \vee L = C \vee L = U$.

Another nice family of partitions

Definition

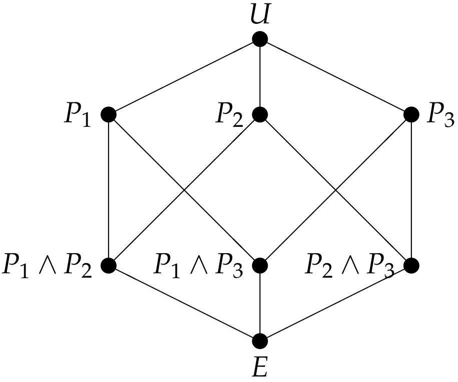
Suppose that P_1, P_2 and P_3 are partitions of a set Ω, none of which is U. Then $\{P_1, P_2, P_3\}$ is a Cartesian decomposition of Ω of dimension 3 if $|\Gamma_1 \cap \Gamma_2 \cap \Gamma_3| = 1$ whenever Γ_i is a part of P_i for $i = 1, 2, 3$.

Another nice family of partitions

Definition

Suppose that P_1, P_2 and P_3 are partitions of a set Ω, none of which is U. Then $\{P_1, P_2, P_3\}$ is a Cartesian decomposition of Ω of dimension 3 if $|\Gamma_1 \cap \Gamma_2 \cap \Gamma_3| = 1$ whenever Γ_i is a part of P_i for $i = 1, 2, 3$.

Taking infima gives a Cartesian lattice.

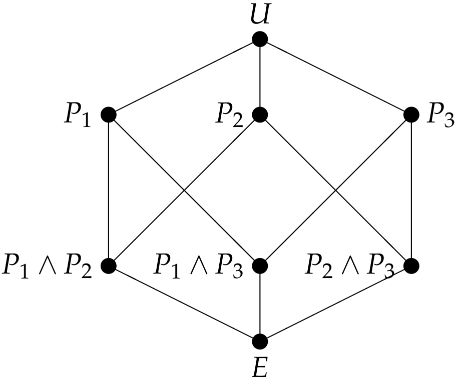


Another nice family of partitions

Definition

Suppose that P_1, P_2 and P_3 are partitions of a set Ω, none of which is U. Then
$\{P_1, P_2, P_3\}$ is a Cartesian decomposition of Ω of dimension 3 if $|\Gamma_1 \cap \Gamma_2 \cap \Gamma_3| = 1$ whenever Γ_i is a part of P_i for $i = 1, 2, 3$.

Taking infima gives a Cartesian lattice.



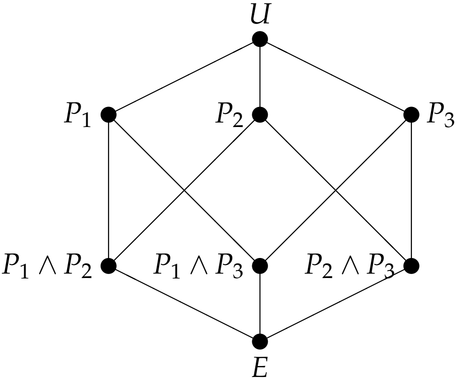
- Each partition is uniform.

Another nice family of partitions

Definition

Suppose that P_1, P_2 and P_3 are partitions of a set Ω, none of which is U. Then
$\{P_1, P_2, P_3\}$ is a Cartesian decomposition of Ω of dimension 3 if $|\Gamma_1 \cap \Gamma_2 \cap \Gamma_3| = 1$ whenever Γ_i is a part of P_i for $i = 1, 2, 3$.

Taking infima gives a Cartesian lattice.



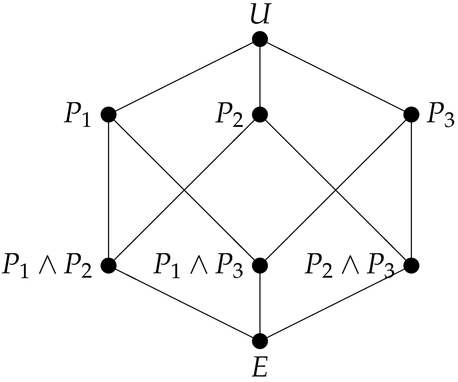
- Each partition is uniform.
- Each pair are compatible.

Another nice family of partitions

Definition

Suppose that P_1, P_2 and P_3 are partitions of a set Ω, none of which is U. Then $\{P_1, P_2, P_3\}$ is a Cartesian decomposition of Ω of dimension 3 if $|\Gamma_1 \cap \Gamma_2 \cap \Gamma_3| = 1$ whenever Γ_i is a part of P_i for $i = 1, 2, 3$.

Taking infima gives a Cartesian lattice.



- Each partition is uniform.
- Each pair are compatible.
- Statisticians call this a completely crossed orthogonal block structure.

Coset partitions

Definition

Let H be a subgroup of a group G. Then P_H is the partition of G into right cosets of H.

Coset partitions

Definition

Let H be a subgroup of a group G. Then P_H is the partition of G into right cosets of H.

Proposition

Let H and K be subgroups of a group G. The following hold.

1. P_H *is uniform.*
2. $P_H \wedge P_K = P_{H \cap K}$.
3. $P_H \vee P_K = P_{\langle H,K \rangle}$.
4. P_H *and* P_K *are compatible if and only if* $HK = KH$.

Orthogonality

Let V be the real vector space $\mathbb{R}^{\Omega}$.

Orthogonality

Let V be the real vector space $\mathbb{R}^{\Omega}$.

If P is any partition of Ω, let V_P be the subspace of V consisting of vectors which are constant on each part of P.

Orthogonality

Let V be the real vector space $\mathbb{R}^\Omega$.

If P is any partition of Ω, let V_P be the subspace of V consisting of vectors which are constant on each part of P.

$$\dim(V_P) = \text{number of parts of } P.$$

Orthogonality

Let V be the real vector space $\mathbb{R}^\Omega$.

If P is any partition of Ω, let V_P be the subspace of V consisting of vectors which are constant on each part of P.

$$\dim(V_P) = \text{number of parts of } P.$$
$$P \preccurlyeq Q \iff V_Q \leq V_P.$$

Orthogonality

Let V be the real vector space $\mathbb{R}^\Omega$.

If P is any partition of Ω, let V_P be the subspace of V consisting of vectors which are constant on each part of P.

$$\dim(V_P) = \text{number of parts of } P.$$
$$P \preccurlyeq Q \iff V_Q \le V_P.$$

In particular, $V_E = V$
and V_U is the 1-dimensional subspace of constant vectors.

Orthogonality

Let V be the real vector space $\mathbb{R}^\Omega$.

If P is any partition of Ω, let V_P be the subspace of V consisting of vectors which are constant on each part of P.

$$\dim(V_P) = \text{number of parts of } P.$$
$$P \preccurlyeq Q \iff V_Q \leq V_P.$$

In particular, $V_E = V$
and V_U is the 1-dimensional subspace of constant vectors.

$$V_P \cap V_Q = V_{P \vee Q}.$$

Orthogonality

Let V be the real vector space $\mathbb{R}^{\Omega}$.

If P is any partition of Ω, let V_P be the subspace of V consisting of vectors which are constant on each part of P.

$$\dim(V_P) = \text{number of parts of } P.$$
$$P \preccurlyeq Q \iff V_Q \le V_P.$$

In particular, $V_E = V$
and V_U is the 1-dimensional subspace of constant vectors.

$$V_P \cap V_Q = V_{P \vee Q}.$$

Consider the standard inner product on V.
Because $V_P \cap V_Q \neq \{\mathbf{0}\}$,
the subspaces V_P and V_Q cannot be orthogonal to each other.

Orthogonality

Let V be the real vector space $\mathbb{R}^{\Omega}$.

If P is any partition of Ω, let V_P be the subspace of V consisting of vectors which are constant on each part of P.

$$\dim(V_P) = \text{number of parts of } P.$$
$$P \preccurlyeq Q \iff V_Q \leq V_P.$$

In particular, $V_E = V$
and V_U is the 1-dimensional subspace of constant vectors.

$$V_P \cap V_Q = V_{P \vee Q}.$$

Consider the standard inner product on V.
Because $V_P \cap V_Q \neq \{\mathbf{0}\}$,
the subspaces V_P and V_Q cannot be orthogonal to each other.

Theorem

If P and Q are uniform and compatible then $V_P \cap V_{P\vee Q}^{\perp}$ is orthogonal to $V_Q \cap V_{P\vee Q}^{\perp}$.

Orthogonal decomposition

Theorem

Suppose that $\mathcal{P}$ is a join semi-lattice of pairwise compatible uniform partitions of Ω. For P in $\mathcal{P}$, put

$$W_P = V_P \cap \left(\sum_{P \prec Q} V_Q \right)^{\perp}.$$

Then the W-subspaces are pairwise orthogonal and

$$V_Q = \bigoplus_{Q \preccurlyeq P} W_P.$$

Orthogonal decomposition

Theorem

Suppose that $\mathcal{P}$ is a join semi-lattice of pairwise compatible uniform partitions of Ω. For P in $\mathcal{P}$, put

$$W_P = V_P \cap \left(\sum_{P \prec Q} V_Q \right)^{\perp}.$$

Then the W-subspaces are pairwise orthogonal and

$$V_Q = \bigoplus_{Q \preccurlyeq P} W_P.$$

The partial order $\preccurlyeq$ has a zeta-function ζ defined by

$$\zeta(Q,P) = \begin{cases} 1 & \text{if } Q \preccurlyeq P, \\ 0 & \text{otherwise.} \end{cases}$$

Orthogonal decomposition

Theorem

Suppose that $\mathcal{P}$ is a join semi-lattice of pairwise compatible uniform partitions of Ω. For P in $\mathcal{P}$, put

$$W_P = V_P \cap \left(\sum_{P \prec Q} V_Q \right)^{\perp}.$$

Then the W-subspaces are pairwise orthogonal and

$$V_Q = \bigoplus_{Q \preccurlyeq P} W_P.$$

The partial order $\preccurlyeq$ has a zeta-function ζ defined by

$$\zeta(Q,P) = \begin{cases} 1 & \text{if } Q \preccurlyeq P, \\ 0 & \text{otherwise.} \end{cases}$$

So $\dim(V_Q) = \sum_P \zeta(Q,P) \dim(W_P)$.

Möbius inversion

$$\boxed{\dim(V_Q) = \sum_P \zeta(Q,P) \dim(W_P)}$$

We can write the names of the partitions in order such that Q comes before P if $Q \prec P$.

Möbius inversion

$$\boxed{\dim(V_Q) = \sum_P \zeta(Q,P) \dim(W_P)}$$

We can write the names of the partitions in order such that Q comes before P if $Q \prec P$.
Then the square matrix ζ is upper-triangular with all entries on the main diagonal equal to 1.

Möbius inversion

$$\boxed{\dim(V_Q) = \sum_P \zeta(Q,P)\dim(W_P)}$$

We can write the names of the partitions in order such that
Q comes before P if $Q \prec P$.
Then the square matrix ζ is upper-triangular
with all entries on the main diagonal equal to 1.
Hence the matrix ζ has an inverse matrix μ,
which is also upper-triangular
with all entries on the main diagonal equal to 1.

Möbius inversion

$$\boxed{\dim(V_Q) = \sum_P \zeta(Q,P)\dim(W_P)}$$

We can write the names of the partitions in order such that Q comes before P if $Q \prec P$.
Then the square matrix ζ is upper-triangular with all entries on the main diagonal equal to 1.
Hence the matrix ζ has an inverse matrix μ, which is also upper-triangular with all entries on the main diagonal equal to 1.
This is called the Möbius function, which was extensively studied by Gian-Carlo Rota.

Möbius inversion

$$\boxed{\dim(V_Q) = \sum_P \zeta(Q,P) \dim(W_P)}$$

We can write the names of the partitions in order such that
Q comes before P if $Q \prec P$.
Then the square matrix ζ is upper-triangular
with all entries on the main diagonal equal to 1.
Hence the matrix ζ has an inverse matrix μ,
which is also upper-triangular
with all entries on the main diagonal equal to 1.
This is called the Möbius function,
which was extensively studied by Gian-Carlo Rota.
Applying so-called Möbius inversion to the equation at the top of this slide gives

$$\dim(W_P) = \sum_Q \mu(P,Q) \dim(V_Q).$$

Chapter 2

Some statistical history

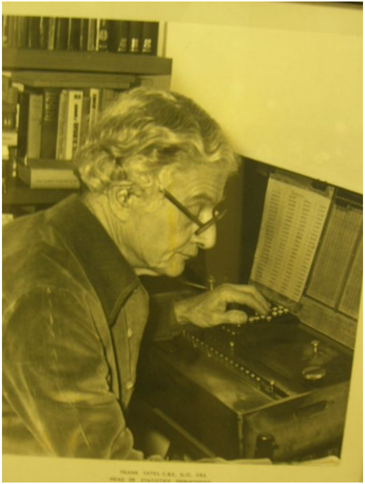
Statisticians at Rothamsted

Here are some of the statisticians who have worked at the agricultural research station at Rothamsted.

Ronald Fisher	1919–1933	then UCL, then Cambridge
Frank Yates	1931–1968	
Oscar Kempthorne	1941–1946	then Ames, Iowa
Desmond Patterson	1947–1967	then Edinburgh
John Nelder	1968–1984	previously National Vegetable Research Station
Rosemary Bailey	1981–1990	
Robin Thompson	1997–2012 (?)	previously Edinburgh

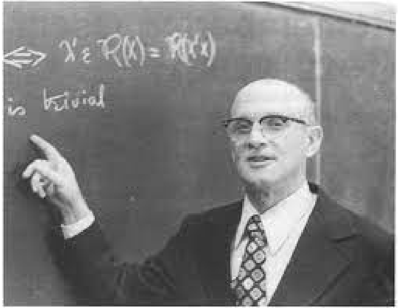
Photos: Fisher and Yates

Ronald Fisher



Frank Yates

Photos: Kempthorne and Patterson



Oscar Kempthorne

Desmond Patterson

Photo: Nelder

John Nelder

Nelder's papers

In 1976–1978 I was employed as a post-doctoral research fellow in the Statistics Department at Edinburgh University.
The aim was to apply ideas from combinatorics and group theory to design of experiments.

Nelder's papers

In 1976–1978 I was employed as a post-doctoral research fellow in the Statistics Department at Edinburgh University.
The aim was to apply ideas from combinatorics and group theory to design of experiments.

At the start, Desmond Patterson gave me copies of John Nelder's two 1965 papers on orthogonal block structure, and told me to read them.

Nelder's papers

In 1976–1978 I was employed as a post-doctoral research fellow in the Statistics Department at Edinburgh University.
The aim was to apply ideas from combinatorics and group theory to design of experiments.

At the start, Desmond Patterson gave me copies of John Nelder's two 1965 papers on orthogonal block structure, and told me to read them.

After three months, I said "OK, I understand them now."

Nelder's papers

In 1976–1978 I was employed as a post-doctoral research fellow in the Statistics Department at Edinburgh University.
The aim was to apply ideas from combinatorics and group theory to design of experiments.

At the start, Desmond Patterson gave me copies of John Nelder's two 1965 papers on orthogonal block structure, and told me to read them.

After three months, I said "OK, I understand them now."

Desmond responded "Hmph! That's good. No one else does."

Nelder's papers

In 1976–1978 I was employed as a post-doctoral research fellow in the Statistics Department at Edinburgh University.
The aim was to apply ideas from combinatorics and group theory to design of experiments.

At the start, Desmond Patterson gave me copies of John Nelder's two 1965 papers on orthogonal block structure, and told me to read them.

After three months, I said "OK, I understand them now."

Desmond responded "Hmph! That's good. No one else does."

I did not believe him then, but, looking back, I can see that his approach did not incorporate Nelder's ideas until much later.

Kempthorne's papers

Then my colleague Robin Thompson gave me a 1961 technical report (long, but in typescript) by Oscar Kempthorne and his colleagues in Ames. This developed essentially the same ideas as Nelder's: lattices of partitions using some of the partitions in a Cartesian lattice (not necessarily with all coordinates having the same number of values, for example, the rows and columns of a rectangle).

Kempthorne's papers

Then my colleague Robin Thompson gave me a 1961 technical report (long, but in typescript) by Oscar Kempthorne and his colleagues in Ames. This developed essentially the same ideas as Nelder's: lattices of partitions using some of the partitions in a Cartesian lattice (not necessarily with all coordinates having the same number of values, for example, the rows and columns of a rectangle).

Later I learnt that Kempthorne was furious that Nelder had "stolen" his ideas. I believe that they simply developed them independently, building on the work of Fisher and Yates. In those days, it took much longer for ideas to circulate widely.

Putting the bits together

One morning, I came into work after drinking too much in the pub the previous evening. I realised that my brain was not capable of serious work, so I gave it the apparently simple task of matching Nelder's block structures with those of Kempthorne. Slowly, I worked through dimensions 1, 2 and 3.

Putting the bits together

One morning, I came into work after drinking too much in the pub the previous evening. I realised that my brain was not capable of serious work, so I gave it the apparently simple task of matching Nelder's block structures with those of Kempthorne. Slowly, I worked through dimensions 1, 2 and 3.

At the end of the day, I hit a problem.
For dimension 4, Nelder's approach gave 15 possibilities, but Kempthorne's gave 16. I gave up and went home.

Putting the bits together

One morning, I came into work after drinking too much in the pub the previous evening. I realised that my brain was not capable of serious work, so I gave it the apparently simple task of matching Nelder's block structures with those of Kempthorne. Slowly, I worked through dimensions 1, 2 and 3.

At the end of the day, I hit a problem.
For dimension 4, Nelder's approach gave 15 possibilities, but Kempthorne's gave 16. I gave up and went home.

The next day, with a clear head, I realised that Kempthorne's approach always gives more possibilities than Nelder's in dimensions at least 4.

Meeting Kempthorne

I worked on these ideas for some years, partly in collaboration with Terry Speed. We used the Möbius function in some formulae.

Meeting Kempthorne

I worked on these ideas for some years, partly in collaboration with Terry Speed. We used the Möbius function in some formulae.

In June 1988 I attended a two-week research workshop at the Institute for Mathematics and its Applications in Minneapolis, USA. At the weekend, another participant, Jonathan Smith, took me to Ames, so that I could have some meetings with Kempthorne.

Meeting Kempthorne

I worked on these ideas for some years, partly in collaboration with Terry Speed. We used the Möbius function in some formulae.

In June 1988 I attended a two-week research workshop at the Institute for Mathematics and its Applications in Minneapolis, USA. At the weekend, another participant, Jonathan Smith, took me to Ames, so that I could have some meetings with Kempthorne. Kempthorne was very friendly, and said that he much appreciated my work, but

> *"This Möbius function really does the job. I wish that we had known about it."*

Chapter 3

Diagonal semilattices

Starting work on diagonal structures

In 2018, Peter Cameron, Cheryl Praeger, Csaba Schneider and I were in Shenzen, China, for a conference dedicated to Cheryl's 70-th birthday. After the conference, CEP and CS showed us something that they were working on that they thought would interest us.

Starting work on diagonal structures

In 2018, Peter Cameron, Cheryl Praeger, Csaba Schneider and I were in Shenzen, China, for a conference dedicated to Cheryl's 70-th birthday. After the conference, CEP and CS showed us something that they were working on that they thought would interest us.

CEP and CS: This is about diagonal groups, permutation groups and Cartesian decompositions.

Starting work on diagonal structures

In 2018, Peter Cameron, Cheryl Praeger, Csaba Schneider and I were in Shenzen, China, for a conference dedicated to Cheryl's 70-th birthday. After the conference, CEP and CS showed us something that they were working on that they thought would interest us.

CEP and CS: This is about diagonal groups, permutation groups and Cartesian decompositions.

PJC: I think it is about Hamming graphs.

Starting work on diagonal structures

In 2018, Peter Cameron, Cheryl Praeger, Csaba Schneider and I were in Shenzen, China, for a conference dedicated to Cheryl's 70-th birthday. After the conference, CEP and CS showed us something that they were working on that they thought would interest us.

CEP and CS: This is about diagonal groups, permutation groups and Cartesian decompositions.

PJC: I think it is about Hamming graphs.

RAB: I think it is about orthogonal block structures.

Starting work on diagonal structures

In 2018, Peter Cameron, Cheryl Praeger, Csaba Schneider and I were in Shenzen, China, for a conference dedicated to Cheryl's 70-th birthday. After the conference, CEP and CS showed us something that they were working on that they thought would interest us.

CEP and CS: This is about diagonal groups, permutation groups and Cartesian decompositions.

PJC: I think it is about Hamming graphs.

RAB: I think it is about orthogonal block structures.

We started to collaborate, and two years later proved a lovely theorem.

Theorem about diagonal semilattices

Theorem about diagonal semilattices

Theorem

Let $\mathcal{Q}$ be a set of $m+1$ partitions of the same set Ω, where $m \geq 2$. Suppose that every subset of m of the partitions in $\mathcal{Q}$ form the minimal non-trivial partitions in a Cartesian lattice of dimension m.

Theorem about diagonal semilattices

Theorem

Let $\mathcal{Q}$ be a set of $m+1$ partitions of the same set Ω, where $m \geq 2$. Suppose that every subset of m of the partitions in $\mathcal{Q}$ form the minimal non-trivial partitions in a Cartesian lattice of dimension m.

(a) *If $m = 2$ then there is a Latin square on Ω, unique up to paratopism, such that $\mathcal{Q} = \{R, C, L\}$.*

A paratopism is any combination of permuting rows, permuting columns, permuting symbols, and interchanging the three partitions amongst themselves.

Theorem about diagonal semilattices

Theorem

Let $\mathcal{Q}$ be a set of $m+1$ partitions of the same set Ω, where $m \geq 2$. Suppose that every subset of m of the partitions in $\mathcal{Q}$ form the minimal non-trivial partitions in a Cartesian lattice of dimension m.

(a) *If $m = 2$ then there is a Latin square on Ω, unique up to paratopism, such that $\mathcal{Q} = \{R, C, L\}$.*

(b) *If $m > 2$ then there is a group G, unique up to group isomorphism, such that Ω may be identified with G^m and the partitions in $\mathcal{Q}$ are the right-coset partitions of the subgroups $G_1, \ldots, G_m, \delta(G)$, where G_i has j-th entry 1 for all $j \neq i$, and $\delta(G)$ is the diagonal subgroup $\{(g, g, \ldots, g) : g \in G\}$.*

A paratopism is any combination of permuting rows, permuting columns, permuting symbols, and interchanging the three partitions amongst themselves.

Theorem about diagonal semilattices

Theorem

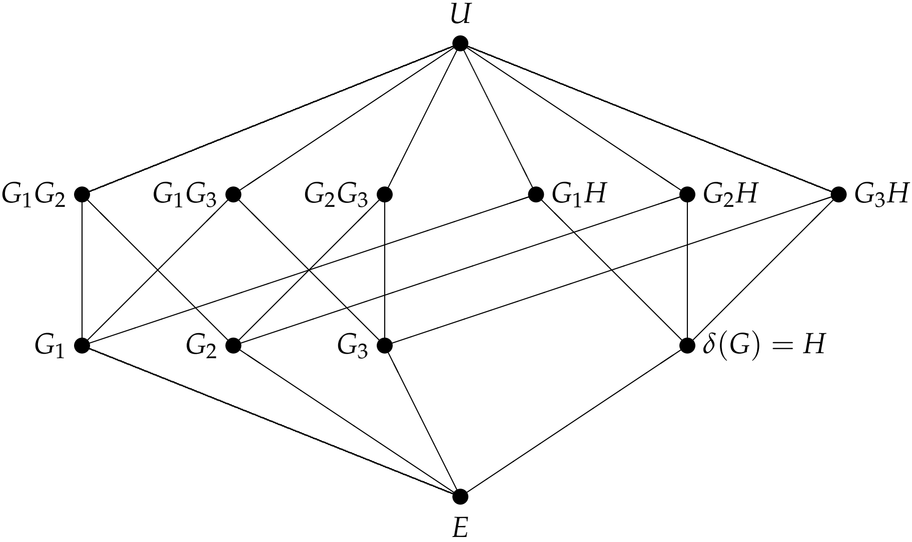
Let $\mathcal{Q}$ be a set of $m+1$ partitions of the same set Ω, where $m \geq 2$. Suppose that every subset of m of the partitions in $\mathcal{Q}$ form the minimal non-trivial partitions in a Cartesian lattice of dimension m.

(a) *If $m = 2$ then there is a Latin square on Ω, unique up to paratopism, such that $\mathcal{Q} = \{R, C, L\}$.*

(b) *If $m > 2$ then there is a group G, unique up to group isomorphism, such that Ω may be identified with G^m and the partitions in $\mathcal{Q}$ are the right-coset partitions of the subgroups $G_1, \ldots, G_m$, $\delta(G)$, where G_i has j-th entry 1 for all $j \neq i$, and $\delta(G)$ is the diagonal subgroup $\{(g, g, \ldots, g) : g \in G\}$.*

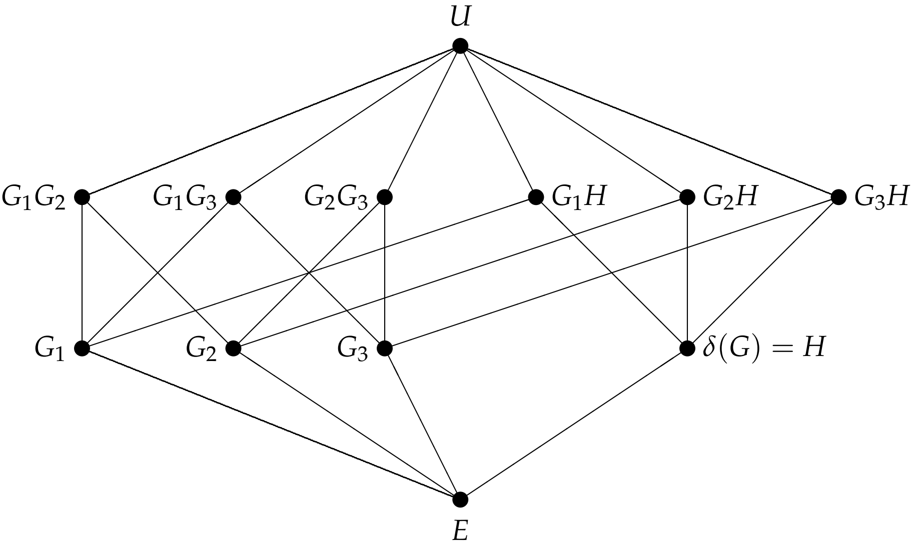
A paratopism is any combination of permuting rows, permuting columns, permuting symbols, and interchanging the three partitions amongst themselves.
For $m > 2$, the combinatorial assumptions in the statement of the theorem force the existence of a group.

Hasse diagram for coset partitions in dimension 3



Hasse diagram for coset partitions in dimension 3

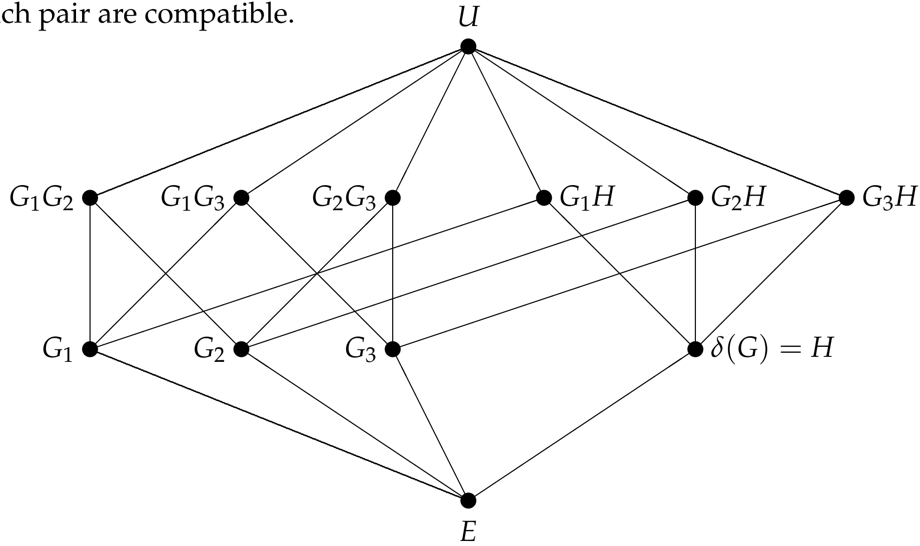
Each partition is uniform.



Hasse diagram for coset partitions in dimension 3

Each partition is uniform.

Each pair are compatible.

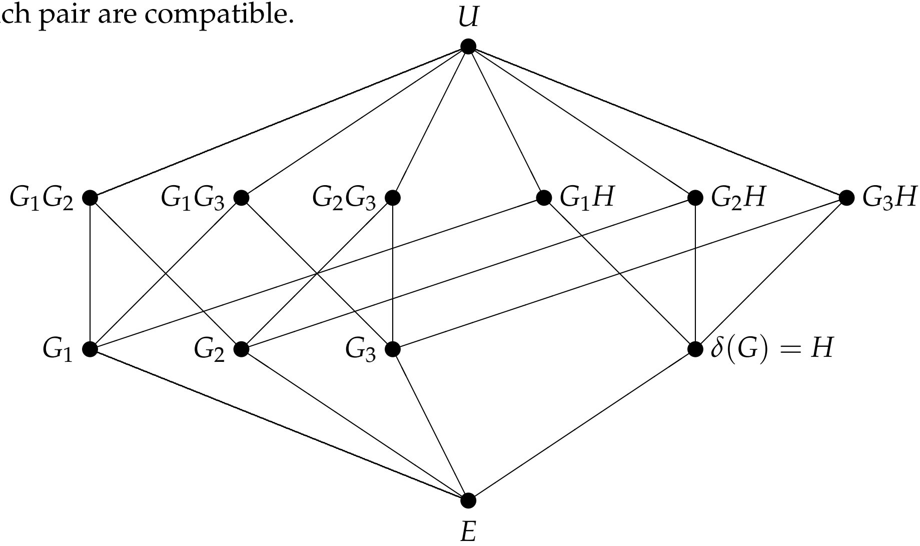


Hasse diagram for coset partitions in dimension 3

Each partition is uniform.

Each pair are compatible.

All suprema are included,



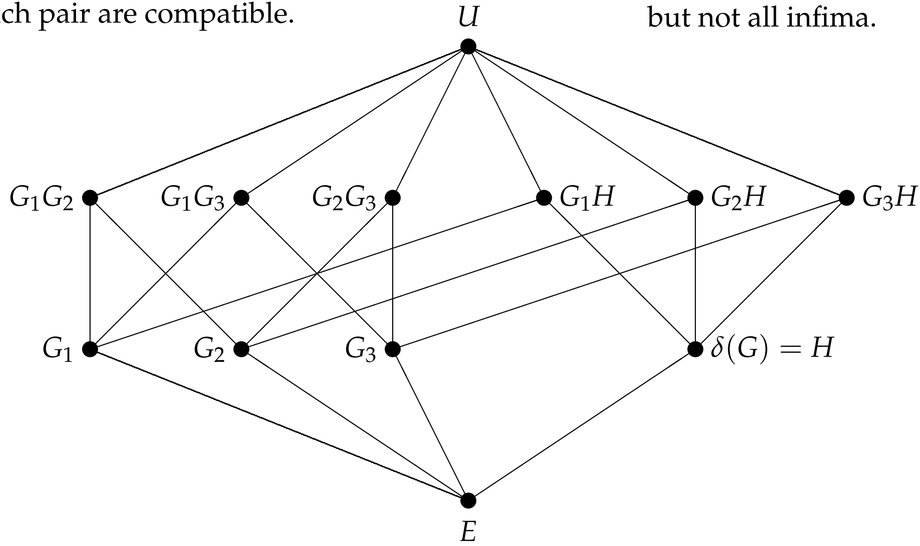
Hasse diagram for coset partitions in dimension 3

Each partition is uniform.

Each pair are compatible.

All suprema are included,

but not all infima.



Comments

1. If the group G is not Abelian, then we cannot include all infima without destroying compatibility.

Comments

1. If the group G is not Abelian, then we cannot include all infima without destroying compatibility.
2. In 1984, Danish statistician Tue Tjur pointed out that, for statistical purposes, closure under suprema is more important than closure under infima,
   and that such closure does not destroy compatibility.

Chapter 4

Diagonal graphs

Hamming graphs

The *Hamming graph* $H(m,n)$ has vertex set A^m,
where A is a set of size n with $n > 1$;
two vertices are joined if they differ in exactly one coordinate.

Hamming graphs

The *Hamming graph* $H(m, n)$ has vertex set A^m,
where A is a set of size n with $n > 1$;
two vertices are joined if they differ in exactly one coordinate.

Here is another way to think about this. The coordinates define the minimal partitions in a Cartesian lattice. Two vertices are joined if they are in the same part of any one of the minimal partitions.

Hamming graphs

The *Hamming graph* $H(m, n)$ has vertex set A^m,
where A is a set of size n with $n > 1$;
two vertices are joined if they differ in exactly one coordinate.

Here is another way to think about this. The coordinates define the minimal partitions in a Cartesian lattice. Two vertices are joined if they are in the same part of any one of the minimal partitions.

When $n = 2$, the Hamming graph can be thought of as the m-dimensional cube.

Hamming graphs

The *Hamming graph* $H(m, n)$ has vertex set A^m,
where A is a set of size n with $n > 1$;
two vertices are joined if they differ in exactly one coordinate.

Here is another way to think about this. The coordinates define the minimal partitions in a Cartesian lattice. Two vertices are joined if they are in the same part of any one of the minimal partitions.

When $n = 2$, the Hamming graph can be thought of as the m-dimensional cube. Now add an extra edge at each vertex, joining it to the vertex which differs from it in all coordinates. This graph is called the *folded cube*.

Hamming graphs

The *Hamming graph* $H(m, n)$ has vertex set A^m,
where A is a set of size n with $n > 1$;
two vertices are joined if they differ in exactly one coordinate.

Here is another way to think about this. The coordinates define the minimal partitions in a Cartesian lattice. Two vertices are joined if they are in the same part of any one of the minimal partitions.

When $n = 2$, the Hamming graph can be thought of as the m-dimensional cube. Now add an extra edge at each vertex, joining it to the vertex which differs from it in all coordinates.
This graph is called the *folded cube*.
When $m = 4$ it is also called the *Clebsch graph*.

Hamming graphs

The *Hamming graph* $H(m, n)$ has vertex set A^m,
where A is a set of size n with $n > 1$;
two vertices are joined if they differ in exactly one coordinate.

Here is another way to think about this. The coordinates define the minimal partitions in a Cartesian lattice. Two vertices are joined if they are in the same part of any one of the minimal partitions.

When $n = 2$, the Hamming graph can be thought of as the m-dimensional cube. Now add an extra edge at each vertex, joining it to the vertex which differs from it in all coordinates.
This graph is called the *folded cube*.
When $m = 4$ it is also called the *Clebsch graph*.

In recent work, Peter Cameron and I have generalized the folded cube to larger values of n, using a diagonal semi-lattice.

Defining a diagonal graph

Given a group G of order n, the diagonal graph $\Gamma_D(G, m)$ of dimension m has vertex set G^m.

Defining a diagonal graph

Given a group G of order n, the diagonal graph $\Gamma_D(G, m)$ of dimension m has vertex set G^m.

Let $Q_1, \ldots, Q_m$ be the the partitions defined by the appropriate coordinates, and let Q_0 be the coset partition of the diagonal subgroup $\delta(G)$. Two distinct vertices are joined if they are in the same part of any one of these $m+1$ partitions.

Defining a diagonal graph

Given a group G of order n, the diagonal graph $\Gamma_D(G, m)$ of dimension m has vertex set G^m.
Let $Q_1, \ldots, Q_m$ be the the partitions defined by the appropriate coordinates, and let Q_0 be the coset partition of the diagonal subgroup $\delta(G)$. Two distinct vertices are joined if they are in the same part of any one of these $m + 1$ partitions.

If $n = 2$, this is the folded cube.

Defining a diagonal graph

Given a group G of order n, the diagonal graph $\Gamma_D(G,m)$ of dimension m has vertex set G^m.
Let $Q_1, \ldots, Q_m$ be the the partitions defined by the appropriate coordinates, and let Q_0 be the coset partition of the diagonal subgroup $\delta(G)$. Two distinct vertices are joined if they are in the same part of any one of these $m+1$ partitions.

If $n = 2$, this is the folded cube.

If $m = 2$, this is the Latin-square graph defined by the Cayley table of G. This is a well-known strongly regular graph.

Some basic properties of the diagonal graph $\Gamma_D(G, m)$

- There are n^m vertices.

Some basic properties of the diagonal graph $\Gamma_D(G, m)$

- There are n^m vertices.
- The valency is $(m+1)(n-1)$.

Some basic properties of the diagonal graph $\Gamma_D(G, m)$

- There are n^m vertices.
- The valency is $(m+1)(n-1)$.
- Except for $n = m = 2$, the clique number is n.

Some basic properties of the diagonal graph $\Gamma_D(G, m)$

- There are n^m vertices.
- The valency is $(m+1)(n-1)$.
- Except for $n = m = 2$, the clique number is n.
- $\Gamma_D(G_1, m_1) \cong \Gamma_D(G_2, m_2) \iff m_1 = m_2$ and $G_1 \cong G_2$.

Some basic properties of the diagonal graph $\Gamma_D(G,m)$

- There are n^m vertices.
- The valency is $(m+1)(n-1)$.
- Except for $n=m=2$, the clique number is n.
- $\Gamma_D(G_1,m_1) \cong \Gamma_D(G_2,m_2) \iff m_1 = m_2$ and $G_1 \cong G_2$.
- The diameter is equal to

$$m+1-\left\lceil \frac{m+1}{n} \right\rceil,$$

  which is less than or equal to m,
  with equality if and only if $n \geq m+1$.

An example with $m = 3$ and $n = 3$

Let $G = \langle x \rangle$, where $x^3 = 1$.

An example with $m = 3$ and $n = 3$

Let $G = \langle x \rangle$, where $x^3 = 1$.

In G^3, put $a = (x, 1, 1)$, $b = (1, x, 1)$ and $c = (1, 1, x)$.

An example with $m = 3$ and $n = 3$

Let $G = \langle x \rangle$, where $x^3 = 1$.

In G^3, put $a = (x, 1, 1)$, $b = (1, x, 1)$ and $c = (1, 1, x)$.

The diagonal semi-lattice has

minimal partitions	Q_0	Q_1	Q_2	Q_3
cosets of	$\langle abc \rangle$	$\langle a \rangle$	$\langle b \rangle$	$\langle c \rangle$

An example with $m = 3$ and $n = 3$

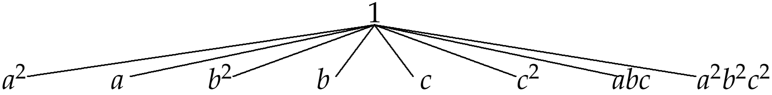
Let $G = \langle x \rangle$, where $x^3 = 1$.

In G^3, put $a = (x, 1, 1)$, $b = (1, x, 1)$ and $c = (1, 1, x)$.

The diagonal semi-lattice has

minimal partitions	Q_0	Q_1	Q_2	Q_3
cosets of	$\langle abc \rangle$	$\langle a \rangle$	$\langle b \rangle$	$\langle c \rangle$

Here are the vertices joined to vertex 1.



An example with $m = 3$ and $n = 3$

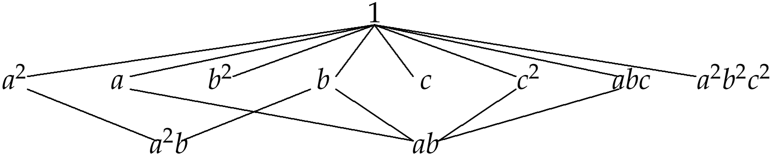
Let $G = \langle x \rangle$, where $x^3 = 1$.

In G^3, put $a = (x, 1, 1)$, $b = (1, x, 1)$ and $c = (1, 1, x)$.

The diagonal semi-lattice has

minimal partitions	Q_0	Q_1	Q_2	Q_3
cosets of	$\langle abc \rangle$	$\langle a \rangle$	$\langle b \rangle$	$\langle c \rangle$

Here are the vertices joined to vertex 1.



Vertices 1 and ab are at distance 2, and have 4 common neighbours. Vertices 1 and a^2b are at distance 2, and have 2 common neighbours.

An example with $m = 3$ and $n = 3$

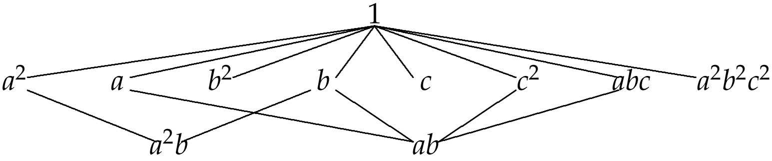
Let $G = \langle x \rangle$, where $x^3 = 1$.

In G^3, put $a = (x, 1, 1)$, $b = (1, x, 1)$ and $c = (1, 1, x)$.

The diagonal semi-lattice has

minimal partitions	Q_0	Q_1	Q_2	Q_3
cosets of	$\langle abc \rangle$	$\langle a \rangle$	$\langle b \rangle$	$\langle c \rangle$

Here are the vertices joined to vertex 1.



Vertices 1 and ab are at distance 2, and have 4 common neighbours. Vertices 1 and a^2b are at distance 2, and have 2 common neighbours. So the graph is not distance-regular.

Eigenvalues of the adjacency matrix

For $i = 0, 1, \ldots, m$, let A_i be the $n \times n$ matrix whose rows and columns are indexed by elements of G with

$$A_i(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \text{ and } \beta \text{ are in the same part of } Q_i \text{ but } \alpha \neq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Eigenvalues of the adjacency matrix

For $i = 0, 1, \ldots, m$, let A_i be the $n \times n$ matrix whose rows and columns are indexed by elements of G with

$$A_i(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \text{ and } \beta \text{ are in the same part of } Q_i \text{ but } \alpha \neq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Then the adjacency matrix A of $\Gamma_D(G, m)$ is given by

$$A = A_0 + A_1 + \cdots + A_m.$$

Eigenvalues of the adjacency matrix

For $i = 0, 1, \ldots, m$, let A_i be the $n \times n$ matrix whose rows and columns are indexed by elements of G with

$$A_i(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \text{ and } \beta \text{ are in the same part of } Q_i \text{ but } \alpha \neq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Then the adjacency matrix A of $\Gamma_D(G, m)$ is given by

$$A = A_0 + A_1 + \cdots + A_m.$$

A_i is the adjacency matrix of a graph which is n^{m-1} disjoint copies of the complete graph K_n, one on each part of Q_i.

Eigenvalues of the adjacency matrix

For $i = 0, 1, \ldots, m$, let A_i be the $n \times n$ matrix whose rows and columns are indexed by elements of G with

$$A_i(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \text{ and } \beta \text{ are in the same part of } Q_i \text{ but } \alpha \neq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Then the adjacency matrix A of $\Gamma_D(G, m)$ is given by

$$A = A_0 + A_1 + \cdots + A_m.$$

A_i is the adjacency matrix of a graph which is n^{m-1} disjoint copies of the complete graph K_n, one on each part of Q_i.
So the eigenvalues of A_i are $n-1$ and -1,
with corresponding eigenspaces V_{Q_i} and $V_{Q_i}^{\perp}$
and corresponding multiplicities n^{m-1} and $n^{m-1}(n-1)$.

Eigenvalues of the adjacency matrix

For $i = 0, 1, \ldots, m$, let A_i be the $n \times n$ matrix whose rows and columns are indexed by elements of G with

$$A_i(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \text{ and } \beta \text{ are in the same part of } Q_i \text{ but } \alpha \neq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Then the adjacency matrix A of $\Gamma_D(G, m)$ is given by

$$A = A_0 + A_1 + \cdots + A_m.$$

A_i is the adjacency matrix of a graph which is n^{m-1} disjoint copies of the complete graph K_n, one on each part of Q_i.
So the eigenvalues of A_i are $n - 1$ and -1,
with corresponding eigenspaces V_{Q_i} and $V_{Q_i}^{\perp}$
and corresponding multiplicities n^{m-1} and $n^{m-1}(n-1)$.
Hence each W-subspace is contained in an eigenspace of A.

Eigenvalues of the adjacency matrix, continued

If Q is a partition in the diagonal semi-lattice,
put $\rho(Q) = k$ if Q is the supremum of exactly k of the minimal partitions $Q_0, Q_1, \ldots, Q_m$. Call $\rho(Q)$ the rank of Q.

Eigenvalues of the adjacency matrix, continued

If Q is a partition in the diagonal semi-lattice,
put $\rho(Q) = k$ if Q is the supremum of exactly k of the minimal partitions $Q_0, Q_1, \ldots, Q_m$. Call $\rho(Q)$ the rank of Q.

If $\rho(Q) = k$ and $\mathbf{v} \in W_Q$ then $\mathbf{v}$ is constant on precisely k of the minimal partitions $Q_0, Q_1, \ldots, Q_m$. Hence the eigenvalue of A on $\mathbf{v}$ is

$$k(n-1) + (m+1-k)(-1) = -(m+1) + kn.$$

Eigenvalues of the adjacency matrix, continued

If Q is a partition in the diagonal semi-lattice, put $\rho(Q) = k$ if Q is the supremum of exactly k of the minimal partitions $Q_0, Q_1, \ldots, Q_m$. Call $\rho(Q)$ the rank of Q.

If $\rho(Q) = k$ and $\mathbf{v} \in W_Q$ then $\mathbf{v}$ is constant on precisely k of the minimal partitions $Q_0, Q_1, \ldots, Q_m$. Hence the eigenvalue of A on $\mathbf{v}$ is

$$k(n-1) + (m+1-k)(-1) = -(m+1) + kn.$$

We know that, for partition P,

$$\dim(V_P) = n^{m-\rho(P)} = \sum_{P \preccurlyeq Q} \dim(W_Q) = \sum_{Q} \zeta(P,Q) \dim(W_Q)$$

Eigenvalues of the adjacency matrix, continued

If Q is a partition in the diagonal semi-lattice,
put $\rho(Q) = k$ if Q is the supremum of exactly k of the minimal partitions $Q_0, Q_1, \ldots, Q_m$. Call $\rho(Q)$ the rank of Q.

If $\rho(Q) = k$ and $\mathbf{v} \in W_Q$ then $\mathbf{v}$ is constant on precisely k of the minimal partitions $Q_0, Q_1, \ldots, Q_m$. Hence the eigenvalue of A on $\mathbf{v}$ is

$$k(n-1) + (m+1-k)(-1) = -(m+1) + kn.$$

We know that, for partition P,

$$\dim(V_P) = n^{m-\rho(P)} = \sum_{P \preccurlyeq Q} \dim(W_Q) = \sum_{Q} \zeta(P,Q) \dim(W_Q)$$

so Möbius inversion gives

$$\dim(W_Q) = \sum_{P} \mu(Q,P) n^{m-\rho(P)}.$$

That Möbius inversion

We managed to prove the following for the diagonal semi-lattice.

Theorem

$$\mu(Q,P) = \begin{cases} (-1)^{\rho(P)-\rho(Q)} & \textit{if } Q \preccurlyeq P \textit{ and } P \neq U, \\ (-1)^{m-\rho(Q)}(m-\rho(Q)) & \textit{if } P = U, \\ 0 & \textit{if } Q \not\preccurlyeq P. \end{cases}$$

That Möbius inversion

We managed to prove the following for the diagonal semi-lattice.

Theorem

$$\mu(Q,P) = \begin{cases} (-1)^{\rho(P)-\rho(Q)} & \textit{if } Q \preccurlyeq P \textit{ and } P \neq U, \\ (-1)^{m-\rho(Q)}(m-\rho(Q)) & \textit{if } P = U, \\ 0 \quad \textit{if } Q \not\preccurlyeq P. \end{cases}$$

Using this gives

$$\dim(W_Q) = n^{-1}(n-1)\left[(n-1)^{m-\rho(Q)} - (-1)^{m-\rho(Q)}\right]$$

if $Q \neq U$, while $\dim(W_U) = 1$.

That Möbius inversion

We managed to prove the following for the diagonal semi-lattice.

Theorem

$$\mu(Q,P) = \begin{cases} (-1)^{\rho(P)-\rho(Q)} & \textit{if } Q \preccurlyeq P \textit{ and } P \neq U, \\ (-1)^{m-\rho(Q)}(m-\rho(Q)) & \textit{if } P = U, \\ 0 & \textit{if } Q \not\preccurlyeq P. \end{cases}$$

Using this gives

$$\dim(W_Q) = n^{-1}(n-1)\left[(n-1)^{m-\rho(Q)} - (-1)^{m-\rho(Q)}\right]$$

if $Q \neq U$, while $\dim(W_U) = 1$.
There are ${}^{m+1}\mathrm{C}_k$ partitions with rank k, if $0 \leq k \leq m-1$, so the eigenvalue $-(m+1)+kn$ has multiplicity

$${}^{m+1}\mathrm{C}_k n^{-1}(n-1)\left[(n-1)^{m-k} - (-1)^{m-k}\right].$$

That Möbius inversion

We managed to prove the following for the diagonal semi-lattice.

Theorem

$$\mu(Q,P) = \begin{cases} (-1)^{\rho(P)-\rho(Q)} & \textit{if } Q \preccurlyeq P \textit{ and } P \neq U, \\ (-1)^{m-\rho(Q)}(m-\rho(Q)) & \textit{if } P = U, \\ 0 & \textit{if } Q \not\preccurlyeq P. \end{cases}$$

Using this gives

$$\dim(W_Q) = n^{-1}(n-1)\left[(n-1)^{m-\rho(Q)} - (-1)^{m-\rho(Q)}\right]$$

if $Q \neq U$, while $\dim(W_U) = 1$.
There are ${}^{m+1}\mathrm{C}_k$ partitions with rank k, if $0 \leq k \leq m-1$, so the eigenvalue $-(m+1)+kn$ has multiplicity

$${}^{m+1}\mathrm{C}_k n^{-1}(n-1)\left[(n-1)^{m-k} - (-1)^{m-k}\right].$$

This just leaves the subspace W_U of constant vectors, which has eigenvalue $(m+1)(n-1)$ with multiplicity 1.

Chapter 5

… and beyond

What next?

In 2020, Peter Cameron, Michael Kinyon, Cheryl Praeger and I started to generalize the previous work to a collection of $m+k$ partitions, with $m \geq 2$ and $k \geq 1$.

What next?

In 2020, Peter Cameron, Michael Kinyon, Cheryl Praeger and I started to generalize the previous work to a collection of $m + k$ partitions, with $m \geq 2$ and $k \geq 1$. Because we have done the case $k = 1$, our assumption now is that $\mathcal{Q}$ is a set of $m + k$ partitions of the same set Ω, where $m \geq 2$ and $k \geq 2$, and that every subset of m of the partitions in $\mathcal{Q}$ form the minimal non-trivial partitions in a Cartesian lattice of dimension m.

What next?

In 2020, Peter Cameron, Michael Kinyon, Cheryl Praeger and I started to generalize the previous work to a collection of $m+k$ partitions, with $m \geq 2$ and $k \geq 1$. Because we have done the case $k=1$, our assumption now is that $\mathcal{Q}$ is a set of $m+k$ partitions of the same set Ω, where $m \geq 2$ and $k \geq 2$, and that every subset of m of the partitions in $\mathcal{Q}$ form the minimal non-trivial partitions in a Cartesian lattice of dimension m.

When $m=2$, this is precisely a collection of k mutually orthogonal Latin squares (MOLS).

What next?

In 2020, Peter Cameron, Michael Kinyon, Cheryl Praeger and I started to generalize the previous work to a collection of $m+k$ partitions, with $m \geq 2$ and $k \geq 1$. Because we have done the case $k=1$, our assumption now is that $\mathcal{Q}$ is a set of $m+k$ partitions of the same set Ω, where $m \geq 2$ and $k \geq 2$, and that every subset of m of the partitions in $\mathcal{Q}$ form the minimal non-trivial partitions in a Cartesian lattice of dimension m.

When $m=2$, this is precisely a collection of k mutually orthogonal Latin squares (MOLS).

Any three of the partitions define a Latin square, so we have ${}^{k+2}\mathrm{C}_3$ such squares.

What next?

In 2020, Peter Cameron, Michael Kinyon, Cheryl Praeger and I started to generalize the previous work to a collection of $m+k$ partitions, with $m \geq 2$ and $k \geq 1$. Because we have done the case $k = 1$, our assumption now is that $\mathcal{Q}$ is a set of $m+k$ partitions of the same set Ω, where $m \geq 2$ and $k \geq 2$, and that every subset of m of the partitions in $\mathcal{Q}$ form the minimal non-trivial partitions in a Cartesian lattice of dimension m.

When $m = 2$, this is precisely a collection of k mutually orthogonal Latin squares (MOLS).

Any three of the partitions define a Latin square, so we have ${}^{k+2}\mathrm{C}_3$ such squares.

We found an interesting example with $k = 2$ and $|\Omega| = 8^2$ where the four Latin squares are all Cayley tables of groups, but those groups come from three different isomorphism classes.

Mutually orthogonal diagonal semilattices

When $m \geq 3$ it is tempting to use a term such as “Latin cube” or “Latin hypercube”, but these have so many different meanings in the literature that we decided on the following definition.

Mutually orthogonal diagonal semilattices

When $m \geq 3$ it is tempting to use a term such as “Latin cube” or “Latin hypercube”, but these have so many different meanings in the literature that we decided on the following definition.

Definition

A set of k mutually orthogonal diagonal semilattices (MODS) of order n is a collection $Q_1, \ldots, Q_{m+k}$ of partitions of a set Ω of size n^m with the property that any m of these partitions are the minimal non-trivial partitions in a Cartesian lattice of dimension m.

Mutually orthogonal diagonal semilattices

When $m \geq 3$ it is tempting to use a term such as "Latin cube" or "Latin hypercube", but these have so many different meanings in the literature that we decided on the following definition.

Definition

A set of k mutually orthogonal diagonal semilattices (MODS) of order n is a collection $Q_1, \ldots, Q_{m+k}$ of partitions of a set Ω of size n^m with the property that any m of these partitions are the minimal non-trivial partitions in a Cartesian lattice of dimension m.

The previous result shows that any subset $\mathcal{S}$ of $m+1$ of these partitions defines a unique group $G_{\mathcal{S}}$ such that the partitions are the right-coset partitions of specified subgroups of $G_{\mathcal{S}}^m$.

Mutually orthogonal diagonal semilattices

When $m \geq 3$ it is tempting to use a term such as "Latin cube" or "Latin hypercube", but these have so many different meanings in the literature that we decided on the following definition.

Definition

A set of k mutually orthogonal diagonal semilattices (MODS) of order n is a collection $Q_1, \ldots, Q_{m+k}$ of partitions of a set Ω of size n^m with the property that any m of these partitions are the minimal non-trivial partitions in a Cartesian lattice of dimension m.

The previous result shows that any subset $\mathcal{S}$ of $m+1$ of these partitions defines a unique group $G_{\mathcal{S}}$ such that the partitions are the right-coset partitions of specified subgroups of $G_{\mathcal{S}}^m$.

It seems obvious that the isomorphism type of $G_{\mathcal{S}}$ should not depend on $\mathcal{S}$, but we have not been able to prove this yet.

Regular mutually orthogonal diagonal semilattices

Let us call a set of MODS regular if the isomorphism type of $G_{\mathcal{S}}$ does not depend on $\mathcal{S}$.

Theorem

If $m \geq 3$ and $k \geq 2$ then the unique (up to isomorphism) group G defined by a regular set of MODS is Abelian. Furthermore, G admits three fixed-point-free automorphisms whose product is the identity.

Orthogonal arrays

If we reverse the partial order of refinement of partitions,
we get a dual concept.

Orthogonal arrays

If we reverse the partial order of refinement of partitions,
we get a dual concept. Now the property is:
any m of these partitions are the maximal non-trivial partitions
in a Cartesian lattice of dimension m.

Orthogonal arrays

If we reverse the partial order of refinement of partitions, we get a dual concept. Now the property is: any m of these partitions are the maximal non-trivial partitions in a Cartesian lattice of dimension m.

This is precisely the definition of an orthogonal array of strength m and index 1, a concept which has been studied by many people.

Orthogonal arrays

If we reverse the partial order of refinement of partitions, we get a dual concept. Now the property is: any m of these partitions are the maximal non-trivial partitions in a Cartesian lattice of dimension m.

This is precisely the definition of an orthogonal array of strength m and index 1, a concept which has been studied by many people.

One way of construcing orthogonal arrays uses elementary Abelian groups.

Orthogonal arrays

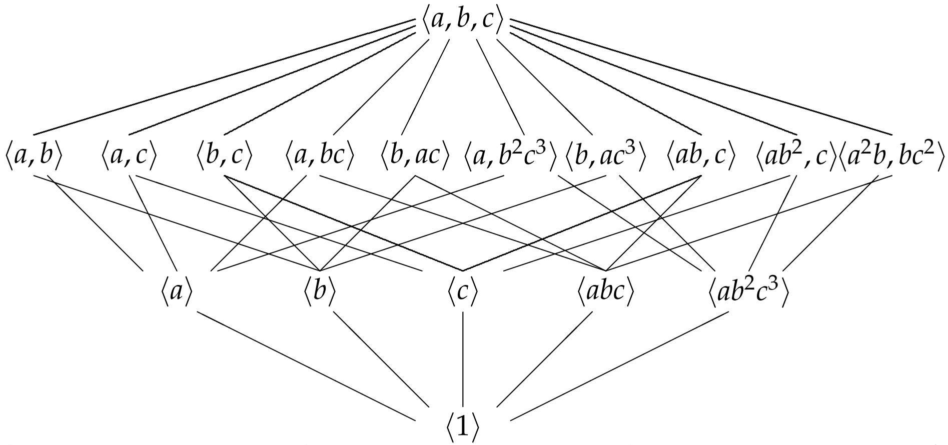
If we reverse the partial order of refinement of partitions, we get a dual concept. Now the property is: any m of these partitions are the maximal non-trivial partitions in a Cartesian lattice of dimension m.

This is precisely the definition of an orthogonal array of strength m and index 1, a concept which has been studied by many people.

One way of construcing orthogonal arrays uses elementary Abelian groups. Taking the dual of such a group (in the algebraic sense) gives the dual concept in the partition sense, which is what we want.

Some subgroups of an elementary Abelian group

If p is prime and $p \geq 5$ we can make a MODS with $n = p^3$, $m = 3$ and $k = 2$ by using some subgroups of an elementary Abelian group of order p^3.



Another MODS

If p is prime and $p \geq 5$ we can make a MODS with $n = p^4$, $m = 4$ and $k = 2$ by using some subgroups of an elementary Abelian group of order p^4.
If $G = \langle a, b, c, d \rangle$ then the six subgroups

$$\langle a \rangle,\ \langle b \rangle,\ \langle c \rangle,\ \langle d \rangle,\ \langle abcd \rangle,\ \langle ab^2c^3d^4 \rangle$$

give the minimal partitions, any four of which generate a Cartesian lattice by taking suprema.

Another MODS

If p is prime and $p \geq 5$ we can make a MODS with $n = p^4$, $m = 4$ and $k = 2$ by using some subgroups of an elementary Abelian group of order p^4.
If $G = \langle a, b, c, d \rangle$ then the six subgroups

$$\langle a \rangle, \ \langle b \rangle, \ \langle c \rangle, \ \langle d \rangle, \ \langle abcd \rangle, \ \langle ab^2c^3d^4 \rangle$$

give the minimal partitions, any four of which generate a Cartesian lattice by taking suprema.

Unfortunately, my slide is too narrow to contain the Hasse diagram.

MODS to graphs

We can construct a graph from a set of MODS in just the same way that we do from a diagonal semi-lattice.

MODS to graphs

We can construct a graph from a set of MODS in just the same way that we do from a diagonal semi-lattice.

The vertices are the elements of the underlying set,
and two distinct vertices are joined by an edge if they are in the same part of any of the minimal non-trivial partitions.

MODS to graphs

We can construct a graph from a set of MODS in just the same way that we do from a diagonal semi-lattice.

The vertices are the elements of the underlying set,
and two distinct vertices are joined by an edge if they are in the same part of any of the minimal non-trivial partitions.

Eigenvalues, and their multiplicities, can be calculated in a similar way as before.

MODS to graphs

We can construct a graph from a set of MODS in just the same way that we do from a diagonal semi-lattice.

The vertices are the elements of the underlying set,
and two distinct vertices are joined by an edge if they are in the same part of any of the minimal non-trivial partitions.

Eigenvalues, and their multiplicities, can be calculated in a similar way as before.

We can use these results to obtain an upper bound for k.

MODS to graphs

We can construct a graph from a set of MODS in just the same way that we do from a diagonal semi-lattice.

The vertices are the elements of the underlying set,
and two distinct vertices are joined by an edge if they are in the same part of any of the minimal non-trivial partitions.

Eigenvalues, and their multiplicities, can be calculated in a similar way as before.

We can use these results to obtain an upper bound for k.

Theorem

Let $m \geq 2$ and $n \geq 2$. If there is a set of MODS of dimension m with $m + k$ minimal non-trivial partitions on a set Ω of size n^m, then $k \leq n - 1$.

MODS to graphs

We can construct a graph from a set of MODS in just the same way that we do from a diagonal semi-lattice.

The vertices are the elements of the underlying set,
and two distinct vertices are joined by an edge if they are in the same part of any of the minimal non-trivial partitions.

Eigenvalues, and their multiplicities, can be calculated in a similar way as before.

We can use these results to obtain an upper bound for k.

Theorem

Let $m \geq 2$ and $n \geq 2$. If there is a set of MODS of dimension m with $m + k$ minimal non-trivial partitions on a set Ω of size n^m, then $k \leq n - 1$.

When $m = 2$, this theorem specializes to the well-known upper bound for the number of mutually orthogonal Latin squares of order n.

References: Partitions in Statistics

- J. A. Nelder: The analysis of randomized experiments with orthogonal block structure. I. Block structure and the null analysis of variance. *Proceedings of the Royal Society of London, Series A* **283** (1965), 147–162.
- J. A. Nelder: The analysis of randomized experiments with orthogonal block structure. II. Treatment structure and the general analysis of variance. *Proceedings of the Royal Society of London, Series A* **283** (1965), 163–178.
- O. Kempthorne, G. Zyskind, S. Addelman, T. N. Throckmorton and R. F. White: *Analysis of Variance Procedures*, Aeronautical Research Laboratory Technical Report 149, Wright–Patterson Air Force Base, Ohio, 1961.
- R. A. Bailey: Orthogonal partitions in designed experiments. *Designs, Codes and Cryptography* **8** (1996), 45–77.
- T. Tjur: Analysis of variance models in orthogonal designs. *International Statistical Review* **52** (1984), 33–81.

References: Diagonal Structures

- R. A. Bailey, Peter J. Cameron, Cheryl E. Praeger and Csaba Schneider: The geometry of diagonal groups. *Transactions of the American Mathematical Society*, in press. doi: 10.1090/tran/8507
- R. A. Bailey, Peter J. Cameron, Michael Kinyon and Cheryl E. Praeger: Diagonal groups and arcs over groups. *Designs, Codes and Cryptography* **108** (2021). doi: 10.1007/s10623-021-00907-2
- R. A. Bailey and Peter J. Cameron: The diagonal graph. *Journal of the Ramanujan Mathematical Society* **36** (2021), 353–361.