The structure of finite groups with restrictions on the set of conjugacy classes sizes

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Definitions

Let G be a finite group,

$Z(G)$ be its center.

$x \in G$, $C_G(x)$ denotes the centralizer of x in G.

x^G denotes the conjugacy class in G containing x.

$Ind(H, x) = |H|/|C_H(x)|$.

$N(G)$ denotes the set of conjugacy classes sizes of G.

William Burnside, 1904

Let p be a prime, G be a finite group. If there exists $x \in G$ such that $Ind(G,x) = p^{\alpha}$, then G is not simple.

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Lev Kazarin, 1990

Let G be a (not necessarily finite) group, $x \in G$ and $Ind(G, x) = p^{\alpha}$, where p is a prime. Then $\langle x^G \rangle$ is a solvable subgroup of G.

Definitions

Let p be a prime, n be a natural number

n_p be the greatest power of p dividing n.

For a set of primes π we will denote $n_\pi = \prod_{p\in\pi} n_p$.

Let $|G||_p$ be a number p^n such that $N(G)$ contains α such that p^n divides α and p^{n+1} does not divide any elements from $N(G)$.

For a set of primes π put $|G||_\pi = \prod_{p\in\pi} |G||_p$.

Definition (I.G. 2022)

Let p be a prime. We say that a group G satisfies the condition $R(p)$ or G is p-index extremal group and write $G \in R(p)$ if $a_p \in \{1, |G|_p\}$ for each $a \in N(G)$ and $|G|_p > 1$.

Examples

$Alt_n \in R(p)$ for each prime $n/2 < p \leq n$.

$L_n(q) \in R(t)$ where t is a primitive prime divisor of $q^n - 1$.

Definition

Let $G \in R(p)$.

(*i*) $G \in R(p)^*$ if G contains a p-element h such that $Ind(G, h)_p > 1$;

(*ii*) $G \in R(p)^{**}$ if $Ind(G, h)_p = 1$ for each p-element $h \in G$.

A. Camina, 1972

If $N(G) = \{1, p^m\} \times \{1, q^n\}$, where p and q are distinct primes, then G is nilpotent, in particular, $G = P \times Q$ for a Sylow p-subgroup P and a Sylow q-subgroup Q.

A. Camina, 1972

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A. Beltran, M.J. Felipe, 2006

If $N(G) = \{1, m\} \times \{1, n\}$, where m and n are positive coprime integers, then G is nilpotent and $n = p^a$ and $m = q^b$ for some distinct primes p and q.

C. Casolo, E. M. Tombari, 2012

Let $p_1^{m_1}, p_2^{m_2}, ..., p_k^{m_k}$ be powers of distinct primes, and let G be a group with $N(G) = \{1, p_1^{m_1}\} \times \{1, p_2^{m_2}\} \times ... \times \{1, p_k^{m_k}\}$. Then G is nilpotent.

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Let $\{p_1^{m_1}, \ldots, p_k^{m_k}\} \subseteq N(G) \subseteq \{1, p_1^{m_1}\} \times \{1, p_2^{m_2}\} \times \ldots \times \{1, p_k^{m_k}\}$. Then G is solvable.

C. Shao, Q. Jiang, 2020

Let further m_1, m_2, m_3 be three positive integers such that m_1 and m_2 do not divide each other and $m_1 m_2$ is coprime to m_3. If $N(G) = \{1, m_1, m_2\} \times \{1, m_3\}$, then $G \simeq A \times B$, where A and B are such that
(a) A is a quasi-Frobenius group with abelian kernel and complement with $N(A) = \{1, m_1, m_2\}$;
(b) $N(B) = \{1, m_3\}$ and m_3 is a prime power.
In particular, G is solvable.

Definition

Given $\Theta \subseteq \mathbb{N}, |\Theta| < \infty$, define the directed graph $\Gamma(\Theta)$, with the vertex set Θ and where $\overrightarrow{ab}$ is an edge whenever a divides b. Put $\Gamma(G) = \Gamma(N(G) \setminus \{1\})$.

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Let Ω be a set of integers and $\Gamma(\Omega \setminus \{1\})$ be disconnected, n be such that $gcd(n, \alpha) = 1$ for each $\alpha \in \Omega \setminus \{1\}$. Let G be a finite group such that $N(G) = \Omega \times \{1, n\}$. Then $G \simeq A \times B$, where $N(A) = \Omega$, $N(B) = \{1, n\}$ and n is a power of prime.

I.G., 2022 (MAIN THEOREM OF THIS TALK)

If $G \in R(p)^*$, then G has a normal p-complement.

Corollaries

I.G., 2022

If $G \in R(p)^*$ and $P \in Syl_p(G)$, then $Z(P) \leq Z(G)$.

A. Vasil'ev, 2009

If G is a group with trivial center and $|G||_p = p$, then Sylow p-subgroups of G are abelian.

I.G., 2022

If $G \in R(p)$ and $Z(G) = 1$, then Sylow p-subgroups of G are abelian.

Sketch of the proof of the main theorem

Let G be a counterexample for assertion of the theorem of minimal order.

Lemma 3.1

$O_{p'}(G) = 1$

Lemma 3.3

Each minimal normal subgroup of G is a p-group.

Proof.
1. $Soc(G) = S \times H$ where S is a simple and H is a p-group.
2. $S \in R(p)^{**}$.
3. By enumeration of all simple groups, I got that S is trivial.

Sketch of the proof of the main theorem

Let $O = O_p(G)$.

Lemmas 3.7 and 3.8

G/O is a simple group and p is a connected component of $GK(G/O)$.

Lemma 3.9

G/O is trivial.

End of proof.

Question

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If $G \in R(p)^{**}$, then G contains at most one non-Abelian composition factor S whose order is divisible by p.

Question 3.

Let $G \in R(p)^{**}$ and S be a non-Abelian composition factor of G whose order is divisible by p. Is it true that G is a semi direct product of a normal subgroup isomorphic to S and some p'-subgroup?