New construction of strongly regular graphs with parameters of symplectic graphs

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Symplectic graph $Sp(2d, q)$

Symplectic graph $Sp(2d, q)$ is the collinearity graph of the symplectic space over F_q of dimension $2d$ according to symplectic form

$$x_1y_2 - x_2y_1 + \cdots + x_{2d-1}y_{2d} - x_{2d}y_{2d-1}.$$

For $d = 2$ the symplectic graph is a $(q+1)$-coclique.

For $d > 2$ this graph is strongly regular, with parameters

$$v = (q^{2d} - 1)/(q - 1), \quad k = q(q^{2d-2} - 1)/(q - 1),$$

$$\lambda = q^2(q^{2d-4} - 1)/(q - 1) + q - 1, \quad \mu = (q^{2d-2} - 1)/(q - 1)).$$

In 2016, Aida Abiad and Willem Haemers used Godsil-McKay switching to obtain strongly regular graphs with the same parameters as $Sp(2d, 2)$ for all $d \geq 3$.

A. Abiad and W.H. Haemers, Switched symplectic graphs and their 2-ranks, Des. Codes Cryptogr., 81 (2016) 35–41.

Introduction

In 2017, Ferdinand Ihringer provided a general construction of strongly regular graphs from the collinearity graph of a finite classical polar spaces of rank at least 3 over a finite field of order q.

F. Ihringer, A switching for all strongly regular collinearity graphs from polar spaces, J. Algebr. Comb., 46 (2017) 263–274.

Recently Andries E. Brouwer, Ferdinand Ihringer and William M. Kantor described a switching operation on collinearity graphs of polar spaces to obtain graphs that satisfy the 4-vertex condition if the original graph belongs to a symplectic polar space.

A. E. Brouwer, F. Ihringer, W. M. Kantor, Strongly regular graphs satisfying the 4-vertex condition, arXiv:2107.00076v1 [math.CO]

Introduction

We present a new construction of strongly regular graphs with parameters

$$v = (q^{2d} - 1)/(q - 1), \quad k = q^{2d-1}, \quad \lambda = \mu = q^{2d-2}(q - 1),$$

which are parameters of the complement of symplectic graphs.

For our construction, we use a new prolific construction of divisible design graphs and do not use switching.

Definition

A k-regular graph on v vertices is a *divisible design graph* with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ if its vertex set can be partitioned into m classes of size n, such that any two distinct vertices from the same class have λ_1 common neighbours, and any two vertices from different classes have λ_2 common neighbours whenever it is not complete or edgeless.

If $m = 1$, then a divisible design graph is strongly regular with parameters (v, k, λ, μ), where $\lambda = \mu = \lambda_1$.

The partition of a divisible design graph into classes is called a *canonical partition*.

Divisible design graphs were first introduced by Willem H. Haemers, Hadi Kharaghani and Maaike Meulenberg.
[HKM] *W.H. Haemers, H. Kharaghani, M. Meulenberg, Divisible design graphs, J. Combinatorial Theory A, 118 (2011) 978–992.*

Wallis-Fon-Der-Flaass construction

In 1971 Walter D. Wallis proposed a new construction of strongly regular graphs based on affine designs and a Steiner 2-design.

In 2002 Dmitriy G. Fon-Der-Flaass found how to modify a partial case of Wallis construction, when the corresponding Steiner 2-design has blocks of size 2, in order to obtain hyperexponentially many strongly regular graphs with the same parameters.

Five years later Misha Muzychuk showed how to modify Fon-Der-Flaass ideas in order to cover all the cases of Wallis construction. Moreover, he showed that a Steiner 2-design in the original Wallis construction can be replaced by a partial linear space and discovered new prolific constructions of strongly regular graphs.

We used some similar ideas to obtain hyperexponentially many divisible design graphs.

Wallis-Fon-Der-Flaass construction

[W] W.D. Wallis, Construction of strongly regular graphs using affine designs, Bull. Austral. Math. Soc. 4 (1971), 41–49.

[FF] D. G. Fon-Der-Flaass New prolific constructions of strongly regular graphs, Adv. Geom. 2 (2002), 301–306.

[MM] M. Muzychuk, A generalization of Wallis-Fon-Der-Flaass construction of strongly regular graphs, J. Algebr. Comb. 25 (2007) 169–187.

Affine designs

Definition

An affine design $D = (V, L)$ is a design, where V is a set of points and L is a set of blocks, with the following two properties:

(i) every two blocks either do not intersect or intersect at a constant number of r points;

(ii) each block together with all blocks disjoint from it forms a parallel class: a set of q mutually disjoint blocks partitioning all points of the design.

Affine designs

If $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is an affine design with parameters q and r, then the number $\varepsilon := (r-1)/(q-1)$ is integer and all parameters of D are expressed in terms of q and r.

v	$q^2 r$	the number of points;
b	$q^3\varepsilon + q^2 + q$	the number of blocks;
m	$q^2\varepsilon + q + 1$	the number of parallel classes;
k	qr	the block size;
λ	$q\varepsilon + 1$	the number of blocks containing any 2 distinct points.

Point-hyperplane affine designs

Any d-dimensional affine space over a finite field of order q with all hyperplanes in the form of blocks is an affine design with $r = q^{d-2}$. This design has q^d points. Any block contains q^{d-1} points, q^{d-2} points are on the intersection of any two different blocks, and the number of blocks containing any 2 distinct points is $(q^{d-1} - 1)/(q - 1)$.

Other known examples of affine designs are Hadamard 3-designs, where $q = 2$.

Definition

A set Q equipped with a binary operation $\circ$ is called a *left quasigroup* if for all elements i and j in Q there is a unique element h such that $i \circ h = j$.

In other words, the mapping $h \mapsto i \circ h$ is a bijection of Q for any $i \in Q$.

In our constructions we use the Cayley table of a left quasigroup.

In the initial version of our construction we used a Latin square instead of a Steiner 2-design.

The author is very grateful to Misha Muzychuk for his valuable remark that it is possible to use the Cayley table of a left quasigroup instead of a Latin square.

Let $\mathcal{D}_1, \ldots, \mathcal{D}_m$ be arbitrary affine designs with parameters (q, q^{d-2}), where $m = (q^d - 1)/(q - 1)$ is the number of parallel classes of blocks in each $\mathcal{D}_i$. Let $\mathcal{D}_i = (\mathcal{P}_i, \mathcal{B}_i)$, for all $i \in [m]$.

Parallel classes in each $\mathcal{D}_i$ are enumerated by integers from $[m]$, and j-th parallel class of $\mathcal{D}_i$ is denoted by $\mathcal{B}_i^j$. If $x \in \mathcal{P}_i$ then the block in $\mathcal{B}_i^j$ which contains x is denoted by $B_i^j(x)$.

Let $\mathcal{Q}$ be a left quasigroup with a binary operation $\circ$ on $[m] := \{1, 2, \ldots, m\}$.

For every pair i, j choose an arbitrary bijection

$$\sigma_{i,j} : \mathcal{B}_i^{i \circ j} \to \mathcal{B}_j^{j \circ i}.$$

We require that $\sigma_{i,j} = \sigma_{j,i}^{-1}$ for all i, j, and $\sigma_{i,i}$ is identity for all i.

Construction 1

Let Γ be a graph defined as follows:

- The vertex set of Γ is $V = \bigcup_{i=1}^{m} \mathcal{P}_i$.
- For any $i, j \in [m]$ two different vertices $x \in \mathcal{P}_i$ and $y \in \mathcal{P}_j$ are adjacent in Γ if and only if $y \notin \sigma_{ij}(B_i^{i \circ j}(x))$.

Theorem 1.

If Γ is a graph from Construction 1, then Γ is a divisible design graph with parameters

$$v = q^d(q^d-1)/(q-1), \quad k = q^{d-1}(q^d-1),$$

$$\lambda_1 = q^{d-1}(q^d-q^{d-1}-1), \quad \lambda_2 = q^{d-2}(q-1)(q^d-1),$$

$$m = (q^d-1)/(q-1), \quad n = q^d.$$

Moreover, Γ has four distinct eigenvalues

$$\{q^d(q^{d-1}-1),\, q^{d-1},\, 0,\, -q^{d-1}\}.$$

Proof of Theorem 1

Let Γ be a graph from Construction 1. The number of vertices in Γ equals $q^d m = q^d(q^d-1)/(q-1)$.
The mapping $j \mapsto i \circ j$ is a bijection of $\mathcal{Q}$. Therefore, if x is a vertex of Γ from $\mathcal{P}_i$, then

$$\Gamma(x) = \bigcup_{j=1}^{m} (\mathcal{P}_j \setminus \sigma_{ij}(B_i^{i\circ j}(x))),$$

and

$$|\mathcal{P}_j \setminus \sigma_{ij}(B_i^{i\circ j}(x))| = |\mathcal{P}_j| - |\sigma_{ij}(B_i^{i\circ j}(x))| = q^d - q^{d-1}.$$

Hence, Γ is a regular graph of degree

$$k = (q^d - q^{d-1})(q^d-1)/(q-1) = q^{d-1}(q^d-1).$$

Proof of Theorem 1

Let x and y be two different vertices from $\mathcal{P}_i$. There are exactly $(q^{d-1}-1)/(q-1)$ blocks in $\mathcal{D}_i$ where x and y appear together. Thus, there are $(q^{d-1}-1)/(q-1)$ cases of j where $\mathcal{D}_j$ $\sigma_{ij}(B_i^{i\circ j}(x))$ and $\sigma_{ij}(B_i^{i\circ j}(y))$ are the same blocks in $\mathcal{D}_j$. Hence, x and y have exactly q^d-q^{d-1} common neighbours in each of these cases.

In all other cases $\sigma_{ij}(B_i^{i\circ j}(x))$ and $\sigma_{ij}(B_i^{i\circ j}(y))$ are disjoint blocks. Hence, x and y have exactly q^d-2q^{d-1} common neighbours in each of the remaining cases.

Therefore, the number of common neighbours for x and y equals

$$(q^d-q^{d-1})\frac{(q^{d-1}-1)}{(q-1)}+(q^d-2q^{d-1})\left(\frac{(q^d-1)}{(q-1)}-\frac{(q^{d-1}-1)}{(q-1)}\right)=$$

$$=q^{d-1}(q^d-q^{d-1}-1).$$

Proof of Theorem 1

Let x be in $\mathcal{P}_i$, and y be in $\mathcal{P}_j$, where $i \neq j$. For each $h \in [m]$ two blocks

$$\sigma_{ih}(B_i^{i\circ h}(x)) \in \mathcal{B}_h^{h\circ i} \quad \text{and} \quad \sigma_{jh}(B_j^{j\circ h}(y)) \in \mathcal{B}_h^{h\circ j}$$

belong to different parallel classes in $\mathcal{D}_h$.

Thus, x and y have exactly $q^d - 2q^{d-1} + q^{d-2}$ common neighbours in each of $m = (q^d - 1)/(q - 1)$ designs.

Hence, the number of common neighbours for x and y equals

$$(q^d - 2q^{d-1} + q^{d-2})\frac{(q^d - 1)}{(q - 1)} = q^{d-2}(q - 1)(q^d - 1)$$

and the partition of vertices Γ with parts on $\mathcal{P}_i$ is a canonical partition. $\square$

Strongly regular graphs with parameters of complement of symplectic graphs

In this section, new strongly regular graphs with parameters

$$\Big((q^{2d}-1)/(q-1), q^{2d-1}, q^{2d-2}(q-1), q^{2d-2}(q-1)\Big)$$

are presented. These parameters are known as parameters of the complement of a symplectic graph $Sp(2d,q)$.

Let Γ^* be a divisible design graph with parameters

$$v^* = q^d(q^d-1)/(q-1), \quad k^* = q^{d-1}(q^d-1),$$

$$\lambda_1^* = q^{d-1}(q^d-q^{d-1}-1), \quad \lambda_2^* = q^{d-2}(q-1)(q^d-1)$$

on the vertex set V^*. The canonical partition of Γ^* consists of $m^* = (q^d-1)/(q-1)$ classes which have the size q^d.

Let $\mathcal{D}^* = (\mathcal{P}^*, \mathcal{B}^*)$ be a symmetric 2-$((q^d-1)/(q-1), q^{d-1}, q^{d-2}(q-1))$ design.

Let ϕ be an arbitrary bijection from the canonical classes of Γ^* to the set of blocks $\mathcal{B}^*$.

Construction 2

Let Γ be a graph defined as follows:

- The vertex set of Γ is $V = V^* \cup \mathcal{P}^*$.
- Two different vertices from V^* are adjacent in Γ if and only if they are adjacent in Γ^*.

  The set $\mathcal{P}^*$ is a coclique in Γ.

  A vertex x in V^* from the canonical class $\mathcal{P}_i$, where $i \in [m^*]$, is adjacent to a vertex y in $\mathcal{P}^*$ if and only if y belongs to the block $\phi(\mathcal{P}_i)$.

Theorem 2

If Γ is a graph from Construction 2, then Γ is a strongly regular graph with parameters

$$((q^{2d}-1)/(q-1), q^{2d-1}, q^{2d-2}(q-1), q^{2d-2}(q-1)).$$

Proof of Theorem 2

Let Γ be a graph from Construction 2. The number of vertices v is equal to

$$q^d(q^d-1)/(q-1)+(q^d-1)/(q-1)=(q^{2d}-1)/(q-1).$$

If x is a vertex of Γ from $\mathcal{P}_i$, where $i \in [m^*]$, then there are

$$k^*+|\phi(\mathcal{P}_i)|=q^{d-1}(q^d-1)+q^{d-1}=q^{2d-1}$$

vertices in $\Gamma(x)$.

If y is a vertex of Γ from $\mathcal{P}^*$, then $\Gamma(y)$ consists of q^{d-1} canonical classes of size q^d from $V \setminus \mathcal{P}^*$.

Hence, Γ is a regular graph of degree

$$k=q^{2d-1}.$$

Proof of Theorem 2

Let x and y be two different vertices from Γ.

There are four different possibilities for mutual arrangement of vertices in Γ:

- $\{x, y\} \subseteq \mathcal{P}_i$, where $i \in [m^*]$;
- $x \in \mathcal{P}_i$, and $y \in \mathcal{P}_j$, where $i, j \in [m^*]$, $i \neq j$;
- $\{x, y\} \subseteq \mathcal{P}^*$;
- $x \in V^*$, and $y \in \mathcal{P}^*$.

Proof of Theorem 2

Proof of Theorem 2

- If $\{x, y\} \subseteq \mathcal{P}_i$ then x and y have $\lambda_1^* = q^{d-1}(q^d - q^{d-1} - 1)$ common neighbours in V^* and q^{d-1} common neighbours in $\mathcal{P}^*$.
- If $x \in \mathcal{P}_i$, and $y \in \mathcal{P}_j$, where $i \neq j$, then x and y have $\lambda_2^* = q^{d-2}(q-1)(q^d - 1)$ common neighbours in V^* and $q^{d-2}(q-1)$ common neighbours in $\mathcal{P}^*$.
- If $\{x, y\} \subseteq \mathcal{P}^*$ then x and y have q^d multiplied by $q^{d-2}(q-1)$ common neighbours in V^* and have no common neighbours in $\mathcal{P}^*$.
- Let $x \in V^*$, and $y \in \mathcal{P}^*$. In this case x has $q^d - q^{d-1}$ common neighbours in any canonical class of Γ^*. Thus, x and y have $q^d - q^{d-1}$ multiplied by q^{d-1} common neighbours in V^* and have no common neighbours in $\mathcal{P}^*$.

Therefore, in all cases the number of common neighbours of x and y in Γ equals $q^{2d-2}(q-1)$. Thus, Γ is a strongly regular graph with parameters

$$((q^{2d}-1)/(q-1), q^{2d-1}, q^{2d-2}(q-1), q^{2d-2}(q-1)).$$

$\square$

For a strongly regular graph Γ with valency k and smallest eigenvalue s, Delsarte proved that the order of a clique in Γ is at most $1-k/s$. This bound is called the *Delsarte bound* and a clique in Γ is a *Delsarte clique* if its order is equal to $1-k/s$. If Γ has parameters of symplectic graph, then $1-k/s=(q^d-1)/(q-1)$. Hence, the number of vertices in $\mathcal{P}^*$ equals Delsarte bound.

Examples with small number of vertices

If $q = 2$ and $d = 2$, then, by Theorem 1, a divisible design graph has parameters $(12, 6, 2, 3, 3, 4)$. The line graph of the octahedron is a unique graph with such parameters.

By Theorem 2, we have the triangular graph $T(6)$ which is a strongly regular graph with parameters $(15, 8, 4, 4)$.

Examples with small number of vertices

Dmitry Panasenko and Leonid Shalaginov found all divisible design graphs up to 39 vertices by direct computer calculations, except for the three tuples of parameters: $(32, 15, 6, 7, 4, 8)$, $(32, 17, 8, 9, 4, 8)$, $(36, 24, 15, 16, 4, 9)$. This cases turned out to be very difficult to calculate.

Their results are available on the web page

`http://alg.imm.uran.ru/dezagraphs/ddgtab.html`

If $q = 3$ and $d = 2$, then, by Theorem 1, we have parameters $(36, 24, 15, 16, 4, 9)$.

By Theorem 2, from any divisible design graph with parameters $(36, 24, 15, 16, 4, 9)$ and a 2-$(4, 3, 2)$ design we can construct a strongly regular graph with parameters $(40, 27, 18, 18)$. The complement of this graph has parameters $(40, 12, 2, 4)$.

All strongly regular graphs with parameters $(40, 12, 2, 4)$ were found by Edward Spence.

E. Spence, The Strongly Regular $(40, 12, 2, 4)$ Graphs, Elec. Journ. Combin., 7(1) (2000) #R22.

Divisible design graphs with parameters $(36, 24, 15, 16, 4, 9)$

There are exactly 28 non-isomorphic strongly regular graphs with parameters $(40, 12, 2, 4)$. Only the first one of them does not have 4-cliques. If strongly regular graph Γ with parameters $(40, 12, 2, 4)$ has a regular 4-clique C, then the induced subgraph on $V(\Gamma) \setminus C$ in the complement of Γ can be a divisible design graph with parameters $(36, 24, 15, 16, 4, 9)$.

Thus, we obtained all non-isomorphic divisible design graphs with parameters $(36, 24, 15, 16, 4, 9)$. Dmitriy Panasenko has calculated all these graphs using Spence's result. It turns out that there are 87 non-isomorphic divisible design graphs with parameters $(36, 24, 15, 16, 4, 9)$.

Questions

Question 1

Is it possible to obtain all non-isomorphic divisible design graphs with parameters

$$v = q^d(q^d - 1)/(q - 1), k = q^{d-1}(q^d - 1),$$

$$\lambda_1 = q^{d-1}(q^d - q^{d-1} - 1), \lambda_2 = q^{d-2}(q - 1)(q^d - 1),$$

$$m = (q^d - 1)/(q - 1), \quad n = q^d$$

from Construction 1 for given q and d?

A similar question for strongly regular graphs from Construction 2 has a "no" answer. There is one strongly regular graph with parameters $(40, 12, 2, 4)$ which we cannot obtain by Construction 2.

Questions

Question 2

How many non-isomorphic divisible design graphs and strongly regular graphs we can get by Construction 1 and Construction 2, respectively, for given q and d?

Questions

Using the same arguments as M. Muzychuk in Proposition 3.5. [MM] we obtain a lower bound for the number of non-isomorphic divisible design graphs by Construction 1

$$\frac{(q!)^m}{(q^d m^2)^{q^d m}(q^{d+1}m)^{m-1}},$$

where $m = (q^d - 1)/(q - 1)$.

Let D_1 be the number of non-isomorphic symmetric 2-$\left(\frac{q^d-1}{q-1}, \frac{q^{d-1}-1}{q-1}, \frac{q^{d-2}-1}{q-1}\right)$ designs.

If Γ^* is a divisible design graphs with parameters from Construction 1, then there are at least D_1 pairwise non-isomorphic strongly regular graphs, by Construction 2, with Γ^* as an induced subgraph.

Questions

To obtain the number of isomorphism classes of strongly regular graphs from Construction 2, we need to estimate the number of Delsarte cliques in a graph with the symplectic graph parameters.

For $q \gg 1$, $d \gg 1$ this number is at most $D_2 = q^{2d-1}(q^{d-2})^{q^{d-2}}$.

Hence, the number of isomorphism classes of strongly regular graphs from Construction 2 is at least

$$\frac{D_1(q!)^m}{D_2(q^d m^2)^{q^d m}(q^{d+1}m)^{m-1}}.$$

Question 3

Is it true that the maximum number of Delsarte cliques in graphs with symplectic graph parameters is attained on the symplectic graph?

If $d = 2$, then the answer is "yes", since the symplectic graph in this case is a generalized quadrangle.

THANK YOU!