On Second Maximal Subgroups of Even Order

Long Miao

E-mail:20210088@hhu.edu.cn

Maximal Subgroups

Considering the quantitative characteristics and embedding properties of maximal subgroups, we can obtain different characterizations of groups.

- Wielandt [1] proved that G is nilpotent if and only if every maximal subgroup of G is normal in G.

- If every maximal subgroup of the group G has prime index, then G is supersolvable ([2]).

- Finite groups whose maximal subgroups are all nilpotent are called Schmidt groups ([3]).

- Thompson [4] proved that a finite group G has a nilpotent maximal subgroup of odd order, then G is solvable.

[1] M. Hall, The Theory of Groups, Macmillan, New York, 1959.

[2] H. G. Bray, W. E. Deskins, D. Johnson, et al., Between Nilpotent and Solvable, Polygonal Publishing House, Passaic, 1982.

[3] O. Y. Schmidt, Groups whose subgroups are all special, Mat. Sb., 31(1924), 366－372.

[4] D. J. S. Robinson, A Course in the Theory of Groups, Springer-Verlag, Berlin-New York, 1982.

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Maximal Subgroups

We introduce the definition of *CAP* and some of its promotions.

Definition 1

A subgroup A of a finite group G a CAP-subgroup of G if for any chief factor H/K of G, we have $H \cap A = K \cap A$ or $HA = KA$.

Definition 2

A subgroup H is said to be semi cover-avoiding (SCAP) in a group G if there is a chief series $1 = G_0 < G_1 < \cdots < G_l = G$ such that for every $i = 1, \cdots, l$, either H covers G_j/G_{j-1} or H avoids G_j/G_{j-1}.

Theorem 1

A group G is solvable if and only if there exists a maximal subgroup M of G such that M is a solvable SCAP-subgroup of G[5].

[5] X. Guo, J. Wang, K. P. Shum, On Semi-Cover-Avoiding Maximal Subgroups and Solvability of Finite Groups, Comm. Algebra, 34(9)(2006), 3235-3244.

Maximal Subgroups

In addition, the maximal subgroups have an important influence in non-solvable groups.

Definition 3

Let H be a subgroup of G and A/B be any pd-chief factor of G. We will say that H is a $CAP_{\mathcal{S}_p^*}$-subgroup of G if either $AH = BH$ or $|A \cap H : B \cap H|_p \leq p$.

Theorem 2

A group $G \in \mathcal{S}_p^*$ if and only if M is a $CAP_{\mathcal{S}_p^*}$-subgroup of G for every maximal subgroup M of G[6].

[6] Z. Gao, J. Li, L. Miao, On $CAP_{\mathcal{S}_p^*}$-subgroups of finite groups, Comm. Algebra, 49(2021), 1120 – 1127.

Second Maximal Subgroups

A second maximal subgroup is a maximal subgroup of maximal subgroup, which is a basic concept in finite group theory and has been studied by many scholars.

- Huppert [7] proved that if every second maximal subgroup of a finite group G is normal in G, then G is supersolvable

- Let G be a group and H be a second maximal subgroup of G. If $H = 1$, then G is solvable([8]).

Note that if the group G is non-solvable, then all its second maximal subgroups are non-trivial.

[7] B. Huppert, Normalteiler und maximale untergruppen endlicher gruppen, Math. Z., 60(1954), 409-434.

[8] S. Li, H. Liu, D. Liu, The solvability between finite groups and semi-subnormal-cover-avoidance subgroups, J. Math., 37(6)(2017), 1303－1308.

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Strictly Second Maximal Subgroups

Definition 4

A subgroup H of a group G is called a strictly second maximal subgroup of G if $H \lessdot M$ for all $M \in Max(G, H)$.

Let $Max_2(G)$ denote the set of all second maximal subgroups of a group G, $Max_2^*(G)$ denotes the set of all strictly second maximal subgroups of G.

Meng and Guo [9] investigated the differences between second maximal subgroups and strictly second maximal subgroups. Konovalova, Monakhov and Sokhor [10] gave the examples of groups with $Max_2(G) = Max_2^*(G)$ and some observations on strictly second maximal subgroups.

[9] B. Huppert, Normalteiler und maximale untergruppen endlicher gruppen, Math. Z., 60(1954), 409-434.

[10] S. Li, H. Liu, D. Liu, The solvability between finite groups and semi-subnormal-cover-avoidance subgroups, J. Math., 37(6)(2017), 1303 – 1308.

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S_4

- S_4(24 阶)
  - $M_1 = S_3$(6 阶) $\{(1),(12),(13),(23),(123),(132)\}$
    - $H_{11} = \langle(12)\rangle = \{(1),(12)\}$ — $H_{111} = \{(1)\}$
    - $H_{12} = \langle(13)\rangle = \{(1),(13)\}$ — $H_{121} = \{(1)\}$
    - $H_{13} = \langle(23)\rangle = \{(1),(23)\}$ — $H_{131} = \{(1)\}$
    - $H_{14} = \langle(123)\rangle = \{(1),(123),(132)\}$ — $H_{141} = \{(1)\}$
  - $M_2 = K_4\langle(12)\rangle$(8 阶) $\{(1),(12)(34),(13)(24),(14)(23),(12),(34),(1324),(1423)\}$
    - $H_{21} = K_4 = \{(1),(12)(34),(13)(24),(14)(23)\}$
      - $H_{211} = \{(1),(12)(34)\}$
      - $H_{212} = \{(1),(13)(24)\}$
      - $H_{213} = \{(1),(14)(23)\}$
    - $H_{22} = \langle(1324)\rangle = \{(1),(12)(34),(1324),(1423)\}$ — $H_{221} = \{(1),(12)(34)\}$
    - $H_{23} = \{(1),(12),(34),(12)(34)\}$
      - $H_{231} = \{(1),(12)\}$
      - $H_{232} = \{(1),(34)\}$
      - $H_{233} = \{(1),(12)(34)\}$
  - $M_3 = A_4$(12 阶) $\{(1),(12)(34),(13)(24),(14)(23),(123),(132),(124),(142),(134),(143),(234),(243)\}$
    - $H_{31} = K_4 = \{(1),(12)(34),(13)(24),(14)(23)\}$
      - $H_{311} = \{(1),(12)(34)\}$
      - $H_{312} = \{(1),(13)(24)\}$
      - $H_{313} = \{(1),(14)(23)\}$
    - $H_{32} = \{(1),(123),(132)\}$ — $H_{321} = \{(1)\}$
    - $H_{33} = \{(1),(124),(142)\}$ — $H_{331} = \{(1)\}$
    - $H_{34} = \{(1),(134),(143)\}$ — $H_{341} = \{(1)\}$
    - $H_{35} = \{(1),(234),(243)\}$ — $H_{351} = \{(1)\}$

S_5

- S_5(120 阶)
  - $M_1 = \langle (12),(13),(14)\rangle \cong S_4$(24 阶)
    - $H_{11} = \langle (123),(124)\rangle \cong A_4$(12 阶)
      - $H_{111} = \langle (123)\rangle \cong Z_3$
      - $H_{112} \cong K_4$(4 阶)
    - $H_{12} = K_4\langle (12)\rangle$(8 阶)
      - $H_{121} \cong K_4$
      - $H_{122} = \langle (12),(34)\rangle \cong Z_2 \times Z_2$
      - $H_{123} = \langle (1324)\rangle \cong Z_4$
    - $H_{13} = \langle (12),(13)\rangle \cong S_3$(6 阶)
      - $H_{131} = \langle (12)\rangle \cong Z_2$
      - $H_{132} = \langle (123)\rangle \cong Z_3$
  - $M_2 = \langle (123),(124),(125)\rangle \cong A_5$(60 阶)
    - $H_{21} = \langle (123),(124)\rangle \cong A_4$(12 阶)(同上)
    - $H_{22} = \langle (12345),(12)(35)\rangle \cong Z_5 : Z_2$(10 阶)
      - $H_{221} = \langle (12345)\rangle \cong Z_5$
      - $H_{222} = \langle (12)(35)\rangle \cong Z_2$
    - $H_{23} = \langle (123),(12)(45)\rangle \cong Z_3 : Z_2$(6 阶)
      - $H_{231} = \langle (123)\rangle \cong Z_3$
      - $H_{232} = \langle (12)(45)\rangle \cong Z_2$
  - $M_3 = \langle (12),(13),(45)\rangle \cong S_3 : Z_2$(12 阶)
    - $H_{31} = \langle (12),(13)\rangle \cong S_3$(同上)
    - $H_{32} = \langle (123)(45)\rangle \cong Z_6$(6 阶)
      - $H_{321} = \langle (123)\rangle \cong Z_3$
      - $H_{322} = \langle (45)\rangle \cong Z_2$
    - $H_{33} = \langle (12),(45)\rangle \cong Z_2 \times Z_2$(4 阶)
      - $H_{331} = \langle (12)\rangle \cong Z_2$
  - $M_4 = \langle (12345),(1243)\rangle \cong Z_5 : Z_4$(20 阶)
    - $H_{41} = \langle (12345),(14)(23)\rangle \cong Z_5 : Z_2$ (同上)
    - $H_{42} = \langle (1243)\rangle \cong Z_4$(4 阶)
      - $H_{421} = \langle (14)(23)\rangle \cong Z_2$

EMN-groups

Recently, Deng, Meng and Lu [11] mainly investigated the structure of *EMN*-groups.

Definition 5

A group G is called to be EMN-group if all maximal subgroups of G of even order are nilpotent.

Theorem 3

Let G be a non-nilpotent EMN-group of even order. Then G is solvable, $|\pi(G)| \leq 3$ *and one of the following statements is true:*
(a) G is a minimal non-nilpotent group;
(b) $G \cong Z_2 \times M$*, where M is a minimal non-nilpotent group of odd order.*

[11] Y. Deng, W. Meng and J. Lu, Finite Groups with Nilpotent Subgroups of Even Order, Bull. Iran. Math. Soc., 48(2022), 1143 – 1152.

Non-nilpotent maximal subgroups of even order

Then we introduce the following set $\mathfrak{X}$.

$$\mathfrak{X} = \{\, M \lessdot G \mid 2 \mid |M| \ and\ M\ is\ non-nilpotent\}.$$

Obviously, if $\mathfrak{X} = \emptyset$, then G is solvable by *Theorem* 3. Correspondingly, we give a new classification of second maximal subgroups by using the core of subgroups.

• $T(\mathfrak{X}) = \{\, H \mid H \lessdot M, where\ M \in \mathfrak{X}\}$;

• $T_1(\mathfrak{X}) = \{\, H \mid H \in T(\mathfrak{X}), \forall M \in Max(G, H)\ s.t.\ H_G = M_G\}$;

• $T_2(\mathfrak{X}) = \{\, H \mid H \in T(\mathfrak{X}), \exists\, M_1 \in Max(G, H)\ s.t.\ H_G = (M_1)_G\ and\ \exists\, M_2 \in Max(G, H)\ s.t.\ H_G < (M_2)_G\}$;

• $T_3(\mathfrak{X}) = \{\, H \mid H \in T(\mathfrak{X}), \forall M \in Max(G, H)\ s.t.\ H_G < M_G\}$;

$\mathcal{F}_{pr}$

Moreover, Guo, Kondrat'ev et al.([12]) describe the non-abelian composition factors of a non-solvable groups class $\mathcal{F}_{pr}$ are pairwise isomorphic and are exhausted by the following groups:

$\mathcal{F}_{pr} = \{ G | M \lessdot G, M \text{ is solvable or the index of } M \text{ in } G \text{ is prime power}\}$

- (1) $PSL_2(2^p)$, where p is a prime;
  (2) $PSL_2(3^p)$, where p is a prime;
  (3) $PSL_2(p^{2^w})$, where p is an odd prime and $w \geq 0$;
  (4) $S_z(2^p)$, where p is an odd prime;
  (5) $PSL_3(3)$.

Here, we define a groups class $\mathcal{F}_p$ containing the groups class $\mathcal{F}_{pr}$

$\mathcal{F}_p = \{ G | non-abelian\ chief\ factor\ H/K\ is\ the\ direct\ product\ of\ the\ simple\ groups\ mentioned\ above\}$

[12] W. Guo, A. S. Kondrat' ev, N. V. Maslova and L. Miao, Finite groups whose maximal subgroups are solvable or have prime power indices, Proc. Steklov Inst. Math., 309(Suppl.1)(2020), 47-51.

$\mathcal{F}_{pr}$

According to the above, we first give characterizations of the class $\mathcal{F}_{pr}$ in terms of the properties of some second maximal subgroups belonging to $T_{12}(\mathfrak{X})$.

$$\begin{aligned} T_{12}(\mathfrak{X}) &= T_1(\mathfrak{X}) \cup T_2(\mathfrak{X}) \\ &= \{ H \mid H \in T(\mathfrak{X}), \exists M \in Max(G, H) \ s.t. \ H_G = M_G \}. \end{aligned}$$

Theorem 4

If $T_{12}(\mathfrak{X}) = \emptyset$, *then* $G \in \mathcal{F}_{pr}$.

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$\mathcal{F}_{pr}$

Theorem 5

If H is a CAP-subgroup of G for every subgroup $H \in T_{12}(\mathfrak{X})$, then $G \in \mathcal{F}_{pr}$.

Theorem 6

If H is a nilpotent subgroup of G for every subgroup $H \in T_{12}(\mathfrak{X})$, then $G \in \mathcal{F}_{pr}$.

In 2022, Wang and Miao [13] introduced a new class of groups $S_p^\#$ containing every group G whose every chief factor H/K satisfies the following condition:

$$\Phi((H/K)_p) = 1$$

- saturated formation, extension closed.
- $G = A_6, p = 3$. $(A_6)_3 = C_3 \times C_3, A_6 \in S_3^\#$.

[13] Y. Wang, L. Miao, Z. Gao and W.Liu, The influence of second maximal subgroups on the generalized p-solvability of finite groups, Comm. Algebra, 50(6)(2022), 2584 – 2591.

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$\mathcal{S}_p^{\#}$

Recently, we use strict second maximal subgroups ($U_1(G)$) to give a characterization of the group class $\mathcal{S}_p^{\#}$.

• $Max_2^*(G) = \{H \mid H \lessdot \lessdot G \text{ and } H \text{ is strict}\}$

• $U_1(G) = T(\mathfrak{X}) \cap Max_2^*(G)$.

Theorem 7

If $U_1(G) = \emptyset$, then G is solvable.

Theorem 8

If H is a CAP-subgroup of G for every subgroup $H \in U_1(G)$, then $G \in \mathcal{S}_2^{\#}$.

$\mathcal{S}_p^{\#}$

In addition, we consider the existence of strict second maximal subgroups of even order, give their properties, and give a characterization of the class of $\mathcal{S}_p^{\#}$.

- $Max_2^2(G) = \{H \mid 2 \mid |H| \text{ and } H \lessdot \lessdot G\}$;
- $U_2(G) = T(\mathfrak{X}) \cap Max_2^*(G) \cap Max_2^2(G)$.

Theorem 9

If $U_2(G) = \emptyset$, then G is solvable.

Theorem 10

If H is a CAP-subgroup of G for every subgroup $H \in U_2(G)$, then $G \in \mathcal{S}_2^{\#}$.

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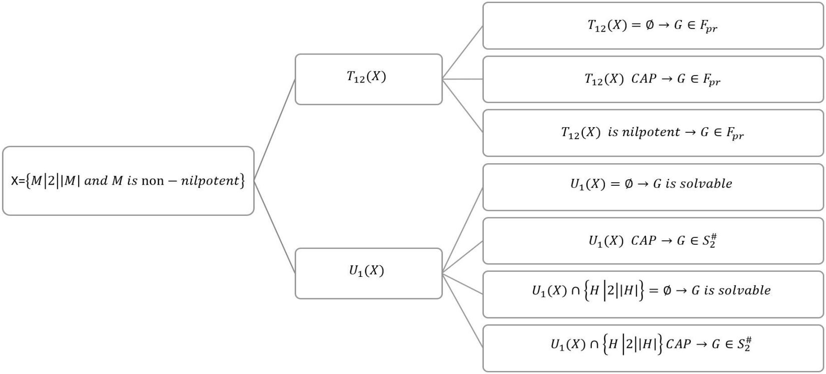
$S_p^{\#}$


X=$\{M \mid 2 \mid |M|$ and M is non $-$ nilpotent$\}$
$T_{12}(X)$
$U_1(X)$
$T_{12}(X) = \emptyset \rightarrow G \in F_{pr}$
$T_{12}(X)$ CAP $\rightarrow G \in F_{pr}$
$T_{12}(X)$ is nilpotent $\rightarrow G \in F_{pr}$
$U_1(X) = \emptyset \rightarrow G$ is solvable
$U_1(X)$ CAP $\rightarrow G \in S_2^{\#}$
$U_1(X) \cap \{H \mid 2 \mid |H|\} = \emptyset \rightarrow G$ is solvable
$U_1(X) \cap \{H \mid 2 \mid |H|\}$ CAP $\rightarrow G \in S_2^{\#}$

$M_G \neq 1$

Notice that the core of every maximal subgroup of a simple group is equal to 1. Therefore, it is natural to consider the effect of maximal subgroups with non-trivial cores on the structure of groups, i.e.,

$$\mathcal{A} = \{ M \lessdot G | \, M_G \neq 1 \}.$$

Furthermore, we give the set of second maximal subgroups associated with it.

- $Max_2^2(G) = \{H | \, 2 | \, |H| \text{ and } H \lessdot \lessdot G\}$;
- $T_1(G) = \{ H | \, H \lessdot \lessdot G, \forall M \in Max(G,H) \; s.t. \; H_G = M_G\}$;
- $Max_2^n(G) = \{H | \, H \lessdot \lessdot G \text{ and } H \text{ is nilpotent}\}$;
- $T(\mathcal{A}) = \{ H | \, H \lessdot M, \text{where } M \in \mathcal{A}\}$;
- $E_1(\mathcal{A}) = T(\mathcal{A}) \cap Max_2^2(G)$;
- $E_2(\mathcal{A}) = E_1 \cap Max_2^*(G)$.

$M_G \neq 1$

Based on existence, we describe the structure of groups using different properties of subgroups. For example, relation of the core of subgroups, nilpotent property of subgroups, *CAP*-property of subgroups and so on.

Theorem 11

Let $G/\Phi(G)$ not be a non-abelian simple group. If $E_1(\mathcal{A}) = \emptyset$, then $G \in \mathcal{F}_{pr}$.

Theorem 12

Let $G/\Phi(G)$ not be a non-abelian simple group. If $E_1(\mathcal{A}) \subseteq Max_2^n(G)$, then $G \in \mathcal{F}_{pr}$.

Theorem 13

If $E_1(\mathcal{A}) \cap \overline{Max_2^n(G)} \subseteq T_1(G)$ and $\overline{Max_2^2(G)} \subseteq T_1(G)$, then $G \in \mathcal{F}_{pr}$.

$M_G \neq 1$

By discussing the set E_2 in the same way as above, we obtain some characterizations of the class of group $\mathcal{F}_{pr}$.

Theorem 14

Let $G/\Phi(G)$ not be a non-abelian simple group. If $E_2(\mathcal{A}) = \emptyset$, then $G \in \mathcal{F}_{pr}$.

Theorem 15

Let $G/\Phi(G)$ not be a non-abelian simple group. If $E_2(\mathcal{A}) \subseteq T_1(G)$, then $G \in \mathcal{F}_{pr}$.

$M_G \neq 1$

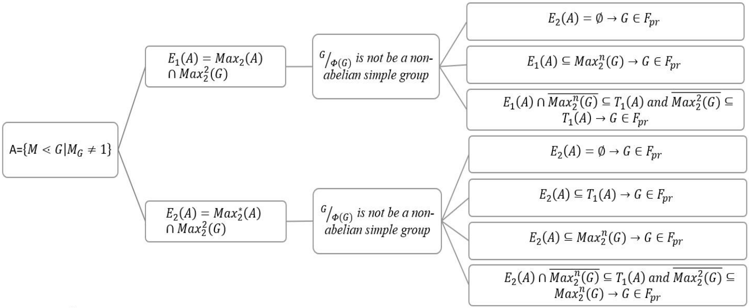
Theorem 16

Let $G/\Phi(G)$ not be a non-abelian simple group. If $E_2(\mathcal{A}) \subseteq Max_2^n(G)$, then $G \in \mathcal{F}_{pr}$.

Theorem 17

If $E_2(\mathcal{A}) \cap \overline{Max_2^n(G)} \subseteq T_1(G)$ and $\overline{Max_2^2(G)} \subseteq Max_2^n(G)$, then $G \in \mathcal{F}_{pr}$.

$M_G \neq 1$



- $Max_2^2(G) = \{H | H << G \text{ and } 2 || H|\}$
- $Max_2^n(G) = \{H | H << G \text{ and } H \text{ is nilpotent}\}$
- $F_{pr} = \{G | M < G, M \text{ is solvable or } |G:M| \text{ is prime power indices}\}$

$p|\ |M|$

Meanwhile, quantitative information is an important way to study the structure of groups. Therefore, we combine the above sets to consider the set of partially maximal subgroups in the dual case, i.e.,

$$\mathcal{B} = \{\, M \lessdot G |\ p|\ |M|\ and\ M_G \neq 1\} \cup \{\, M \lessdot G |\ p \nmid |M|\ and\ 2|\ |M|\}.$$

Furthermore, we give the following families of subgroups for a given group.

•$T_1(G) = \{\, H |\ H \lessdot \lessdot G, \forall M \in Max(G, H)\ s.t.\ H_G = M_G\}$;

•$Max_2^n(G) = \{H |\ H \lessdot \lessdot G\ and\ H\ is\ nilpotent\}$;

•$T(\mathcal{B}) = \{\, H |\ H \lessdot M, where\ M \in \mathcal{B}\}$;

•$Max_2^*(\mathcal{B}) = T(\mathcal{B}) \cap Max_2^*(G)$.

$p \mid |M|$

We first consider the set $T(\mathcal{B})$, and utilize it to study the class of group $\mathcal{F}_{pr}$.

Theorem 18

Let $G/\Phi(G)$ not be a non-abelian simple group. If $T(\mathcal{B}) \subseteq T_1(G)$, then $G \in \mathcal{F}_{pr}$.

Theorem 19

If H is a CAP-subgroup of G for every subgroup $H \in T(\mathcal{B})$, then $G \in \mathcal{F}_{pr}$.

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For the set $Max_2^*(\mathcal{B})$, we first consider its existence. Then we entrust it different properties.

Theorem 20

Let $G/\Phi(G)$ not be a non-abelian simple group. If $Max_2^*(\mathcal{B}) = \emptyset$, then $G \in \mathcal{F}_{pr}$.

Theorem 21

If H is a CAP-subgroup of G for every subgroup $H \in Max_2^*(\mathcal{B})$, then $G \in \mathcal{F}_{pr}$.

Similarly, we also obtain the following results:

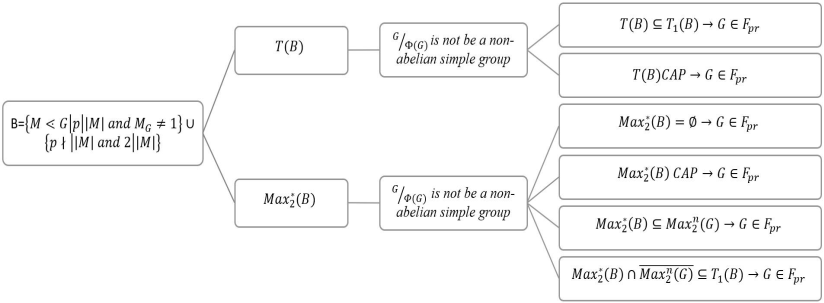
Theorem 22

Let $G/\Phi(G)$ not be a non-abelian simple group. If $Max_2^*(\mathcal{B}) \subseteq Max_2^n(G)$, then $G \in \mathcal{F}_{pr}$.

Theorem 23

Let $G/\Phi(G)$ not be a non-abelian simple group. If $Max_2^*(\mathcal{B}) \cap \overline{Max_2^n(G)} \subseteq T_1(G)$, then $G \in \mathcal{F}_{pr}$.

$p| \ |M|$



- $F_{pr} = \{G | M < G, M \text{ is solvable or } |G:M| \text{ is prime power indices}\}$
- $Max_2^n(G) = \{H | H << G \text{ and } H \text{ is nilpotent}\}$

NS-supplement

Monakhov and Trofimuk [14] introduced the definition of *NS*-supplement.

Definition 6

A subgroup A of a group G is said to be NS-supplemented in G if there exists a subgroup B of G such that:
(1) $G = AB$;
(2) whenever X is a normal subgroup of A and $p \in \pi(B)$, *there exists a Sylow p-subgroup* B_p *of B such that* $XB_p = B_pX$.

Theorem 24

If all maximal subgroups of G are NS-supplemented in G, then G is solvable.

[14] V. S. Monakhov, A. A. Trofimuk, On the supersolubility of a finite group with *NS*-supplemented subgroups, Acta Math. Hungar., 160(1)(2020), 161 – 167.

NS-supplement

We introduce the following families of subgroups for a given group G.

$$U = \{ M \lessdot G | \, M \; is \; not \; NS - supplemented \}.$$

- $Max_2(U) = \{ H | \, H \lessdot M, where \; M \in U \}$;
- $T_{12}(U) = \{ H | \, H \in Max_2(U), \exists M \in Max(G, H) \; s.t. \; H_G = M_G \}$;
- $\overline{Max_2^n(G)} = \{ H | \, H \lessdot \lessdot G \; and \; H \; is \; non - nilpotent \}$;

NS-supplement

We first investigate the influence of the second maximal subgroup on the structure of the group in the set $T_{12}(\mathfrak{X})$.

Theorem 25

If H is NS-supplemented in G for every subgroup $H \in T_{12}(U)$, then $G \in \mathcal{F}_p$.

Theorem 26

If H is a SCAP-subgroup of G for every subgroup $H \in T_{12}(U)$, then $G \in \mathcal{F}_p$.

NS-supplement

Theorem 27

If $T_{12}(U) \cap \overline{Max_2^n(G)} = \emptyset$, *then* $G \in \mathcal{F}_p$.

Theorem 28

If H *is a SCAP-subgroup of* G *for every subgroup* $H \in T_{12}(U) \cap \overline{Max_2^n(G)}$, *then* $G \in \mathcal{F}_p$.

Theorem 29

If H *is a SCAP-subgroup of* G *for every subgroup* $H \in T_{12}(U) \cap \overline{Max_2^n(G)}$, *then* $G \in \mathcal{F}_p$.

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NS-supplement

We have added to the set in the following study.

$$U' = \{ M \lessdot G |\ M \text{ is not } NS-\text{supplemented}\} \cup \{ M \lessdot G |\ M \text{ is } NS-\text{supplemented and } M_G \neq 1\}.$$

- $Max_2^2(G) = \{H |\ 2|\ |H| \text{ and } H \lessdot \lessdot G\}$;
- $E(U) = T(U) \cap Max_2^2(G)$;
- $Max_2^*(U') = Max_2(U') \cap Max_2^*(G)$.

NS-supplement

Theorem 30

If H is a CAP-subgroup of G for every subgroup $H \in E(U)$, then $G \in \mathcal{F}_p$.

Theorem 31

If $E(U) \cap \overline{Max_2^n(G)} = \emptyset$, then $G \in \mathcal{F}_p$.

Theorem 32

If H is a CAP-subgroup of G for every subgroup $H \in E(U') \cap \overline{Max_2^n(G)}$, then $G \in \mathcal{F}_p$.

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NS-supplement

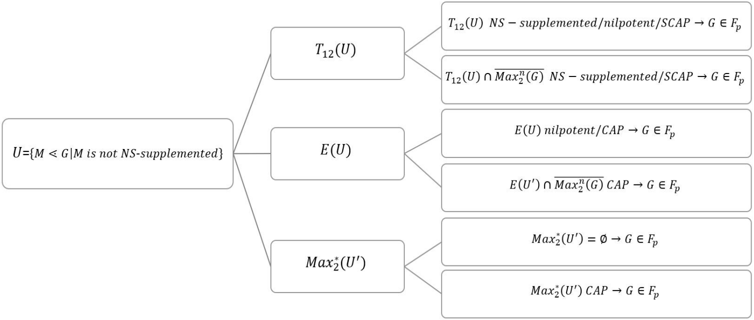
Theorem 33

If $Max_2^*(U') = \emptyset$*, then* $G \in \mathcal{F}_p$.

Theorem 34

If H *is a CAP-subgroup of* G *for every subgroup* $H \in Max_2^*(U')$*, then* $G \in \mathcal{F}_p$.

NS-supplement



- $U' = \{M \lessdot G | M \text{ is not } NS-supplemented\} \cup \{M \lessdot G | M \text{ is } NS-supplemented \text{ and } M_G \neq 1\}$
- $\overline{Max_2^n(G)} = \{H | H \ll G \text{ and } H \text{ is non}-nilpotent\}$

G/M_G

As a result of the above investigations, we have found that the core properties of subgroups and the properties of some subgroups of even order have an important influence on the structure of the group. Inspired by this, we give a new set of maximal subgroups.

$$\overline{C} = \{ M \lessdot G |\ G/M_G \ is\ non-nilpotent\}.$$

Furthermore, we give the set of second maximal subgroups associated with it.

- $Max_2^2(G) = \{H|\ 2|\ |H|\ and\ H \lessdot \lessdot G\}$;
- $Max_2(\overline{C}) = \{H|\ \exists M \in \overline{C}, s.t.\ H \lessdot M\}$;
- $Max_2^*(\overline{C}) = Max_2(\overline{C}) \cap Max_2^*(G)$;
- $T_{12}(\overline{C}) = \{ H|\ H \in Max_2(\overline{C}), \exists M \in Max(G, H)\ s.t.\ H_G = M_G\}$;
- $E(\overline{C}) = T(\overline{C}) \cap Max_2^2(G)$.

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G/M_G

We first consider the existence of these sets and obtain the following relevant characterizations.

Theorem 35

Let G be a group. If $Max_2^*(\overline{C}) = \emptyset$, then G is solvable.

Theorem 36

Let G be a group. If $Max_2^*(\overline{C}) \cap T_{12}(\overline{C}) = \emptyset$, then G is solvable.

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Theorem 37

Let G be a group. If $E(\overline{C}) = \emptyset$, then G is solvable.

Theorem 38

Let G be a group. If $E(\overline{C}) \cap T_{12}(\overline{C}) = \emptyset$, then G is solvable.

Theorem 39

Let G be a group. If $E(\overline{C}) \cap Max_2^*(\overline{C}) = \emptyset$, then $G \in \mathcal{S}_2^\#$.

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G/M_G

Then we assign different properties to them on the basis of their existence. We take the example of the cover-avoidance property of the subgroup.

Theorem 40

Let G be a group. If H is a CAP-subgroup of G for every subgroup $H \in Max_2^*(\overline{C})$, *then G is solvable.*

Theorem 41

Let G be a group. If H is a CAP-subgroup of G for every subgroup $H \in Max_2^*(\overline{C}) \cap T_{12}(\overline{C})$, *then G is solvable.*

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Theorem 42

Let G be a group. If H is a CAP-subgroup of G for every subgroup $H \in E(\overline{C})$, *then G is solvable.*

Theorem 43

Let G be a group. If H is a CAP-subgroup of G for every subgroup $H \in E(\overline{C}) \cap T_{12}(\overline{C})$, *then G is solvable.*

Theorem 44

Let G be a group. If H is a CAP-subgroup of G for every subgroup $H \in E(\overline{C}) \cap Max_2^*(\overline{C})$, *then* $G \in \mathcal{S}_2^{\#}$.

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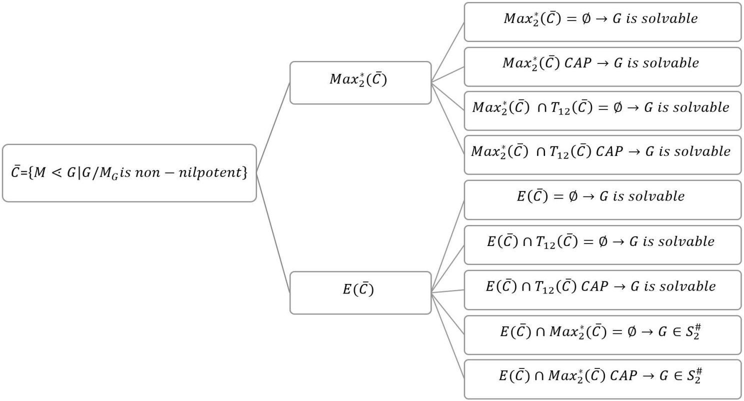
G/M_G



$\bar{C}$={$M < G | G/M_G$ is non − nilpotent}
$Max_2^*(\bar{C})$
$E(\bar{C})$
$Max_2^*(\bar{C}) = \emptyset \rightarrow G$ is solvable
$Max_2^*(\bar{C})$ CAP $\rightarrow G$ is solvable
$Max_2^*(\bar{C}) \cap T_{12}(\bar{C}) = \emptyset \rightarrow G$ is solvable
$Max_2^*(\bar{C}) \cap T_{12}(\bar{C})$ CAP $\rightarrow G$ is solvable
$E(\bar{C}) = \emptyset \rightarrow G$ is solvable
$E(\bar{C}) \cap T_{12}(\bar{C}) = \emptyset \rightarrow G$ is solvable
$E(\bar{C}) \cap T_{12}(\bar{C})$ CAP $\rightarrow G$ is solvable
$E(\bar{C}) \cap Max_2^*(\bar{C}) = \emptyset \rightarrow G \in S_2^\#$
$E(\bar{C}) \cap Max_2^*(\bar{C})$ CAP $\rightarrow G \in S_2^\#$

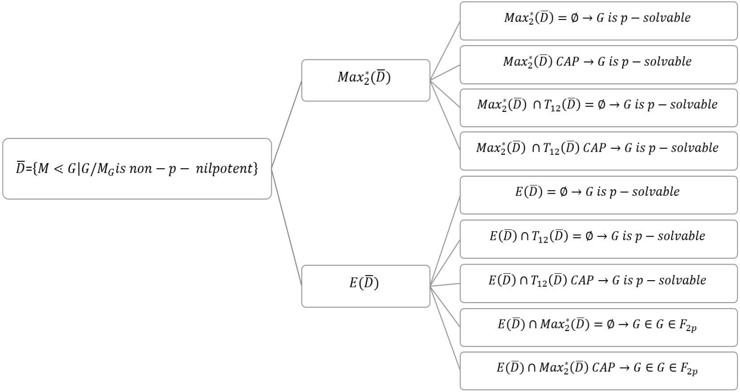
G/M_G

Thus a question can be posed naturally, i.e., if the set $\overline{\mathcal{C}}$ is localized, what is the effect on the structure of the group?

$$\overline{\mathcal{D}} = \{\, M \lessdot G |\, G/M_G \text{ is non} - p - \text{nilpotent}\}.$$

Parallel to the previous results, we obtain the following characterization.

G/M_G



$F_{2p} = \{G | H/K$ is abel or H/K is $p' -$ group or $(H/K)_2$ is abel, where H/K is chief factor of $G\}$

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Thank you!