Detecting properties of a finite group through the study of some functions on element orders

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Basic Motivation

Let G be a periodic group.

Main Problem

To obtain information about the structure of G by looking at the orders of its elements.

The functions $\psi(G)$ and $o(G)$

Let G be a finite group.

Definitions

$$\psi(G) := \sum_{x \in G} o(x).$$

$$o(G) := \frac{1}{|G|} \sum_{x \in G} o(x).$$

Problem

What can be said about the structure of G

by looking at the values $\psi(G)$, $o(G)$?

The functions $\psi(G)$ and $o(G)$

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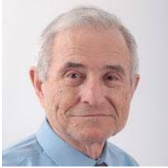
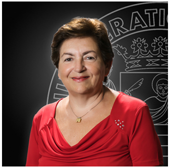


Marcel Herzog
(University of Tel Aviv)



Patrizia Longobardi
(University of Salerno)

Outline of the talk

Let G be a finite group.

- *The function* $\psi(G)$
- *The function* $o(G)$
- *Some other functions related to element orders*

Let G be a finite group.

Definition

$$\psi(G) := \sum_{x \in G} o(x).$$

Remark

$$|G| \leq \psi(G) \leq |G|^2.$$

The function $\psi(G)$

Examples

$$\psi(\mathcal{S}_3) = 13.$$

For, $\psi(\mathcal{S}_3) = 1 \cdot 1 + 3 \cdot 2 + 2 \cdot 3.$

$$\psi(\mathcal{C}_6) = 21.$$

For, $\psi(\mathcal{C}_6) = 1 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + 2 \cdot 6.$

where $\mathcal{C}_n$ is the cyclic group of order n and $\mathcal{S}_3$ is the symmetric group of degree 3.

The function $\psi(G)$

Examples

$$\psi(\mathcal{A}_4) = 31.$$

For, $\psi(\mathcal{A}_4) = 1 \cdot 1 + 3 \cdot 2 + 8 \cdot 3.$

$$\psi(\mathcal{D}_{10}) = 31.$$

For, $\psi(\mathcal{D}_{10}) = 1 \cdot 1 + 5 \cdot 2 + 4 \cdot 5.$

where $\mathcal{A}_n$ is the alternating group of degree n and $\mathcal{D}_n$ is the dihedral group of degree n.

The function $\psi(G)$

Remark

$\psi(G) = \psi(G_1)$ does not imply $G \simeq G_1$.

Example

Let $A = \mathcal{C}_8 \times \mathcal{C}_2$,

$B = \mathcal{C}_2 \ltimes \mathcal{C}_8$, where $\mathcal{C}_2 = \langle a \rangle$, $\mathcal{C}_8 = \langle b \rangle$, $b^a = b^5$.

Then

$\psi(A) = \psi(B) = 87$.

Remark

$|G| = |G_1|$ and $\psi(G) = \psi(G_1)$ do not imply $G \simeq G_1$.

The function $\psi(G)$

Remark

$$\psi(G) = \psi(\mathcal{S}_3) \text{ implies } G \simeq \mathcal{S}_3.$$

$$\psi(G) = \psi(\mathcal{A}_5) \text{ implies } G \simeq \mathcal{A}_5.$$

The function $\psi(G)$

Proposition

If $G = G_1 \times G_2$, then $\psi(G) \leq \psi(G_1)\psi(G_2)$.

If $G = G_1 \times G_2$, where $|G_1|$ and $|G_2|$ are coprime, then $\psi(G) = \psi(G_1)\psi(G_2)$.

The function $\psi(G)$

Remark

$$\psi(C_n) = \sum_{d|n} d\varphi(d),$$

where φ is the Euler's function.

Proposition

Let p be a prime, $\alpha \geq 0$. Then:

$\psi(C_{p^\alpha}) = \frac{p^{2\alpha+1}+1}{p+1}$.

The function $\psi(G)$

Remark

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The function $\psi(G)$

Proposition

Let p be a prime, $\alpha \geq 0$. Then: $\psi(\mathcal{C}_{p^\alpha}) = \frac{p^{2\alpha+1}+1}{p+1}$.

Proof. $\psi(\mathcal{C}_{p^\alpha}) = 1 + p\varphi(p) + p^2\varphi(p^2) + \cdots + p^\alpha(\varphi(p^\alpha)) =$

$1 + p(p-1) + p^2(p^2 - p) + \cdots + p^\alpha(p^\alpha - p^{\alpha-1}) =$

$= 1 + p^2 - p + p^4 - p^3 + \cdots + p^{2\alpha} - p^{2\alpha-1} = \frac{p^{2\alpha+1}+1}{p+1}$, as required.//

Corollary

Let $n > 1$. Write $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, p_i's different primes, α_i's > 0. Then

$$\psi(\mathcal{C}_n) = \prod_{i \in \{1,\cdots,s\}} \frac{p_i^{2\alpha_i+1}+1}{p_i+1}.$$

The function $\psi(G)$

Proposition

Let p be a prime, $\alpha \geq 0$. Then: $\psi(\mathcal{C}_{p^\alpha}) = \frac{p^{2\alpha+1}+1}{p+1}$.

Proof. $\psi(\mathcal{C}_{p^\alpha}) = 1 + p\varphi(p) + p^2\varphi(p^2) + \cdots + p^\alpha(\varphi(p^\alpha)) =$

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Corollary

Let $n > 1$. Write $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, $p_i's$ different primes, α_i's > 0. Then

$$\psi(\mathcal{C}_n) = \prod_{i \in \{1, \cdots, s\}} \frac{p_i^{2\alpha_i+1} + 1}{p_i + 1}.$$

The function $\psi(G)$

Proposition

Let p be a prime, $\alpha \geq 0$. Then: $\psi(\mathcal{C}_{p^\alpha}) = \frac{p^{2\alpha+1}+1}{p+1}$.

Proof. $\psi(\mathcal{C}_{p^\alpha}) = 1 + p\varphi(p) + p^2\varphi(p^2) + \cdots + p^\alpha(\varphi(p^\alpha)) =$
$1 + p(p-1) + p^2(p^2 - p) + \cdots + p^\alpha(p^\alpha - p^{\alpha-1}) =$
$= 1 + p^2 - p + p^4 - p^3 + \cdots + p^{2\alpha} - p^{2\alpha-1} = \frac{p^{2\alpha+1}+1}{p+1}$, as required.//

Corollary

Let $n > 1$. Write $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, p_i's different primes, α_i's > 0. Then

$$\psi(\mathcal{C}_n) = \prod_{i \in \{1,\cdots,s\}} \frac{p_i^{2\alpha_i+1}+1}{p_i+1}.$$

The function $\psi(G)$

Theorem 1 [H. Amiri, S.M. Jafarian Amiri, M. Isaacs, 2009]

Let G be a finite group, $|G| = n$. Then

$$\psi(G) \leq \psi(\mathcal{C}_n).$$

Moreover

$$\psi(G) = \psi(\mathcal{C}_n) \text{ if and only if } G \simeq \mathcal{C}_n.$$

H. Amiri, S.M. Jafarian Amiri, I.M. Isaacs, Sums of element orders in finite groups, *Comm. Algebra* **37** (2009), 2978-2980.

The function $\psi(G)$

Theorem 2 [M. Herzog, P. Longobardi, M. M., 2018]

Let G be a non-cyclic group of order n. Then

$$\psi(G) \leq \frac{7}{11}\psi(C_n).$$

Moreover

this bound is best possible.

M. Herzog, P. Longobardi, M. Maj, An exact upper bound for sums of element orders in non-cyclic finite groups, *J. Pure Appl. Algebra*, **222** n. 7 (2018), 1628-1642.

Remark

The upper bound $\frac{7}{11}$ is best possible.

For example,

$\psi(\mathcal{C}_2 \times \mathcal{C}_2) = 7$ and $\psi(\mathcal{C}_4) = 11$. Therefore

$$\psi(\mathcal{C}_2 \times \mathcal{C}_2) = \frac{7}{11}\psi(\mathcal{C}_4).$$

Moreover,

it is easy to see that if $n = 4k$ for some odd integer k, then

the group $G = \mathcal{C}_{2k} \times \mathcal{C}_2$ satisfies the above equality.

The function $\psi(G)$

Theorem 3 **[M. Herzog, P. Longobardi, M. M., 2021]**

Let G be a non-cyclic group of order n. Then

$$\psi(G) = \frac{7}{11}\psi(C_n)$$

if and only if

$n = 4k$ with $(k, 2) = 1$ and $G = (C_2 \times C_2) \times C_k$.

M. Herzog, P. Longobardi, M. Maj, The second maximal groups with respect to the sum of element orders, *J. Pure Appl. Algebra*, **225** n. 3 (2021), 1-12.

The function $\psi(G)$

Theorem 4 [**M. Herzog, P. Longobardi, M. M., 2021**]

Let G be a non-cyclic group of order n and let q be the smallest prime divisor of n. Then

$$\psi(G) \leq \frac{((q^2-1)q+1)(q+1)}{q^5+1}\psi(C_n)$$

and the equality holds if and only if

$$n = q^2k \text{ with } (k, q!) = 1 \text{ and } G = (C_q \times C_q) \times C_k.$$

M. Herzog, P. Longobardi, M. Maj, The second maximal groups with respect to the sum of element orders, *J. Pure Appl. Algebra*, **225** n. 3 (2021), 1-12.

The function $\psi(G)$

Theorem [M. Herzog, P. Longobardi, M. M.]

Let G be a non-cyclic group of order n and let q be the smallest prime divisor of n. Then

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and the equality holds if and only if

$$n = q^2k \text{ with } (k, q!) = 1 \text{ and } G = (C_q \times C_q) \times C_k.$$

Notice that for $q = 2$ we have:

$$\frac{((2^2-1)2+1)(2+1)}{2^5+1} = \frac{(3 \cdot 2+1)3}{33} = \frac{7}{11}.$$

The function ψ - a solvability criterium

Theorem 5 [M. Herzog, P. Longobardi, M. M., 2018]

Let G be a finite group of order n and suppose that

$$\psi(G) \geq \frac{1}{6.68}\psi(C_n).$$

Then

G is a solvable group.

M. Herzog, P. Longobardi, M. Maj, Two new criteria for solvability of finite groups,, *J. Algebra* **511** (2018), 215-226.

The function ψ - a solvability criterion

Remark

Notice that $\psi(\mathcal{A}_5) = 211$ and $\psi(\mathcal{C}_{60}) = 1617$. Therefore

$$\psi(\mathcal{A}_5) = \frac{211}{1617}\psi(\mathcal{C}_{60}) < \frac{1}{6.68}\psi(\mathcal{C}_{60}).$$

Conjecture

If G is a group of order n and

$$\psi(G) > \frac{211}{1617}\psi(\mathcal{C}_n),$$

then G is solvable.

If true, this lower bound is certainly best possible.

The function ψ - a solvability criterion

Theorem 6 [**M. Baniasad Azad, B. Khosravi, 2018**]

If G is a group of order n and

$$\psi(G) > \frac{211}{1617}\psi(C_n),$$

then G is solvable.

Moreover, if $G = \mathcal{A}_5 \times C_m$, where $(30, m) = 1$,

then $\psi(G) = \frac{211}{1617}\psi(C_n)$.

M. Baniasad Azad, B. Khosravi, A Criterion for Solvability of a Finite Group by the Sum of Element Orders, *J. Algebra* **516** (2018), 115-124.

The function ψ - a solvability criterion

Theorem [A. Bahri, B. Khosravi, Z. Akhlaghi]

If G is a non-solvable group of order n and

$$\psi(G) = \frac{211}{1617}\psi(C_n),$$

then $G = A_5 \times C_m$, where is $(30, m) = 1$.

A. Bahri, B. Khosravi, Z. Akhlaghi, A result on the sum of element orders of a finite group, *Arch. Math.* (Basel) **114** (1) (2020), 3-12.

Theorem 7 [**M. Tărnăuceanu, 2021**]

Let G be a group of order n with $\psi(G) > \frac{13}{21}\psi(\mathcal{C}_n)$.

Then G is nilpotent.

Moreover $\psi(G) = \frac{13}{21}\psi(\mathcal{C}_n)$ if and only if

then $G = \mathcal{S}_3 \times \mathcal{C}_m$, where is $(6, m) = 1$.

M. Tărnăuceanu, A criterion for nilpotency of a finite group by the sum of element orders, *Comm. Algebra* **49** (4) (2021), 1571-1577.

The function ψ - a supersolvability criterion

Theorem 9 [M. Baniasad Azad, B. Khosravi, 2021]

Let G be a group of order n with

$$\psi(G) > \frac{31}{77}\psi(\mathcal{C}_n).$$

Then G is supersolvable.

M. Baniasad Azad, B. Khosravi, On two conjectures about the sim of element orders, *Canadian Math. Bull.* **65** (4) (2021), 30-38.

Proofs - some ingredients

Lemma

If R is a normal cyclic Sylow subgroup of the finite group G, then

$$\psi(G) \leq \psi(R)\psi(G/R),$$

with equality if and only if R is central in G.

Lemma

Let H be a normal subgroup of the finite group G. Then

$$\psi(G) \leq \psi(G/H)|H|^2.$$

Proofs - some ingredients

Theorem [A. Lucchini]

Let A be a cyclic proper subgroup of G

and let $K = core_G(A)$.

Then $[A : K] < [G : A]$

In particular, if $|A| \geq [G : A]$, then $K > 1$.

The function $\psi(G)$ - minimum

Definition

Let n be a positive integer. Put

$$\mathcal{T}_n := \{\psi(H) \mid |H| = n\}$$

$\psi(C_n)$ is the **maximum** of $\mathcal{T}_n$.

Problem

What is the structure of G if $\psi(G)$ is the **minimum** *of $\mathcal{T}_n$?*

The function $\psi(G)$ - minimum

Remarks

If $n = p^\alpha$ for some prime p and some $\alpha > 0$ and $|G| = p^\alpha$,
then obviously
$\psi(G)$ is minimum if and only if $expG = p$.

If $p = 2$ and $\psi(G)$ is minimum ,
then G is the elementary abelian group of order 2^α.

But there are non-isomorphic groups G and G_1 of order $p^\alpha > p^2$ ($p > 2$)
with $\psi(G) = \psi(G_1)$ minimum.

For instance, the two groups of exponent 3 and order 3^3.

The function $\psi(G)$ - minimum

Problem

What happens in the general case?

The function $\psi(G)$ - minimum

Question

If S is a simple group of order n,
is $\psi(S)$ the minimum of $\mathcal{T}_n$?

NO!

There are non-isomorphic simple groups S and S_1 such that $|S| = |S_1|$ and $\psi(S) \neq \psi(S_1)$.

For instance, the groups $\mathcal{A}_8$ and $\mathcal{PSL}(3,4)$ are such that $|\mathcal{A}_8| = 20160 = |\mathcal{PSL}(3,4)|$ and $\psi(\mathcal{A}_8) = 137047 > 103111 = \psi(\mathcal{PSL}(3,4))$.

The function $\psi(G)$ - minimum

Question

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The function $\psi(G)$ - minimum

Question [H. Amiri, S.M. Jafarian Amiri, 2011]

If G is a finite non-simple group and S a finite simple group, $|G| = |S|$.

Is

$$\psi(S) < \psi(G)?$$

NO!

Theorem [Y. Marefat, A. Iranmanesh, A. Tehranian, 2013]

Let $S = \mathcal{PSL}(2, 64)$ and $G = C_{32} \times \mathcal{S}z(8)$.

Then $|G| = |S|$ and $\psi(G) < \psi(S)$.

The function $\psi(G)$ - minimum

Question [H. Amiri, S.M. Jafarian Amiri, 2011]

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Is

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Theorem [Y. Marefat, A. Iranmanesh, A. Tehranian, 2013]

Let $S = \mathcal{PSL}(2, 64)$ and $G = C_{3^2} \times \mathcal{S}z(8)$.

Then $|G| = |S|$ and $\psi(G) < \psi(S)$.

Question

If G is a finite soluble group and S a simple group, $|G| = |S|$.

Is

$$\psi(S) < \psi(G)?$$

NO!

The function $\psi(G)$ - minimum

Question

If G is a finite soluble group and S a simple group, $|G| = |S|$. Is

$$\psi(S) < \psi(G)?$$

NO!

The function $\psi(G)$ - minimum

Theorem [**M. Jahani, Y. Marefat, H. Refaghat, B.V. Fasaghandisi**, 2021]

There exist two finite groups G and S, with S simple and G solvable such that $|G| = |S|$ and $\psi(G) < \psi(S)$.

M. Jahani, Y. Marefat, H. Refaghat, B.V. Fasaghandisi, The minimum sum of element orders of finite groups, *Int. J. Group Theory* **10** (2) (2021), 55-60.

The function $\psi(G)$ - minimum

Theorem [H. Amiri, S.M. Jafarian Amiri, 2011]

Let G be a finite nilpotent group of order n

and assume that there are non-nilpotent groups of order n.

Then there exists a non-nilpotent group K with $|K| = |G|$ such that

$$\psi(K) < \psi(G).$$

The function $o(G)$

Let G be a finite group.

Definition

$$o(G) := \frac{1}{|G|}\sum_{x\in G} o(x).$$

The function $o(G)$

Remark

$$\psi(G) \geq 1 + 2(|G| - 1).$$

Hence,

$$o(G) \geq 2 - \frac{1}{|G|} \geq \frac{3}{2}.$$

If G is an elementary abelian 2-group, then

$$o(G) = 2 - \frac{1}{|G|}.$$

The function $o(G)$

Remark

If G is not an elementary abelian 2-group, then

$$o(G) \geq 2 + \frac{1}{|G|}.$$

In fact, if $x \in G$ with $o(x) > 2$, then also $o(x^{-1}) > 2$ and $x \neq x^{-1}$.

$$\psi(G) \geq 1 + o(x) + o(x^{-1}) + 2(|G| - 3) \geq$$

$$1 + 3 + 3 + 2(|G| - 3) = 2|G| + 1.$$

Then

$$o(G) = \frac{\psi(G)}{|G|} \geq 2 + \frac{1}{|G|}.$$

The function $o(G)$

Remark

$o(G) \leq 2$ if and only if

G is an elementary abelian 2-group and $o(G) = 2 - \frac{1}{|G|}$.

Remark

If $|G|$ is odd, then

$o(G) \geq 3 - \frac{1}{|G|} \geq \frac{7}{3}$.

The function $o(G)$

Remark

If $G = A \times B$, with $(|A|, |B|) = 1$, then

$$o(G) = o(A)o(B).$$

In particular, if $A \neq 1$ and $B \neq 1$, then

$$o(G) \geq \frac{7}{2}.$$

Remark

If G is a finite group and
N a non-trivial normal subgroup of G, then
$$o(G/N) < o(G).$$

The function $o(G)$

Theorem 11 [A. Jaikin-Zapirain, 2011]

Let G be a finite group. Then

$$k(G) \geq o(G) \geq o(Z(G)),$$

where $k(G)$ is the number of the conjugacy classes of G.

A. Jaikin-Zapirain, On the number of conjugacy classes of finite nilpotent groups, *Adv. Math.* **227** (2011), 1129-1143.

The function $o(G)$

Conjecture

Let G be a finite p-group (p a prime)
and let N be a normal (abelian) subgroup of G.

Is it true that $o(G) \geq o(N)^{\frac{1}{2}}$?

A. Jaikin-Zapirain, On the number of conjugacy classes of finite nilpotent groups, *Adv. Math.* **227** (2011), 1129-1143.

The function $o(G)$

The answer is **NO**.

Theorem 12 [E. Khukhro, A. Moretó, M. Zarrin, 2021]

Let $c > 0$ be a real number and let $p \geq \frac{3}{c}$ be a prime. Then there exists a finite p-group with a normal abelian subgroup N such that

$$o(G) < o(N)^c.$$

E.I. Khukhro, A. Moretó, M. Zarrin, The average element order and the number of conjugacy classes of a finite group, *J. Algebra* **569** (2021), 1-11.

The function $o(G)$

Problem

Let G be a finite group such that

$$o(G) < o(\mathcal{A}_5) = \frac{211}{60}.$$

Is G solvable?

E.I. Khukhro, A. Moretó, M. Zarrin, The average element order and the number of conjugacy classes of a finite group, *J. Algebra* **569** (2021), 1-11.

The function $o(G)$ - another solvability criterium

The answer is **YES**.

Theorem 13 [**M. Herzog, P. Longobardi, M.M., 2022**]

Let G be a finite group. Suppose that

$$o(G) \leq o(\mathcal{A}_5).$$

Then either $G \simeq \mathcal{A}_5$ or G is solvable.

M. Herzog, P. Longobardi, M. Maj, On a criterion for solvability of a finite group, *J. Algebra* **597** (2022), 1-23.

The function $o(G)$ - another solvability criterium

Remark

$$o(\mathcal{A}_5) = \frac{211}{60} = 3.51666...$$

Remark

If

$$o(G) < o(\mathcal{A}_5) = \frac{211}{60} = 3.51666...,$$

then G is solvable.

The function $o(G)$

Theorem [M.-S. Lazorec, M. Tărnăuceanu, M. Herzog, P. Longobardi, M.M., 2022]

Let G be a finite group. Suppose that

$$o(G) < o(\mathcal{S}_3).$$

Then G is an elementary abelian 2-group.

Corollary

There are no finite groups G such that

$$o(G) \in [2, \tfrac{13}{6}).$$

M.-S. Lazorec, M. Tărnăuceanu, On the average order of a finite group, *to appear*

Corollary

There are no finite groups G such that

$$o(G) = 2.$$

Proposition

There are no finite groups G such that

$$o(G) = 3.$$

The function $o(G)$

Corollary

There are no finite groups G such that

$$o(G) = 3.$$

Proof

We know that $\psi(G)$ is odd. If $o(G) = 3$, then $|G|$ is odd.

If every element of G has order 3, then

$\psi(G) = 1 + 3(|G| - 1)$ and $o(G) = 3 - \frac{2}{|G|} < 3$.

Therefore there exists an element c of order 5,

all the non-trivial powers of c have order 5 and

$\psi(G) \geq 1 + 4 \cdot 5 + 3(|G| - 5)$ and $o(G) = 3 + \frac{6}{|G|} > 3$.

The function $o(G)$

Definition

$$Imm(o) := \{o(G) \mid G \text{ a finite group}\}$$

Then $2, 3 \notin Imm(o)$. Also any even number is not in $Imm(o)$.

Problem

Are there integer values in $Imm(o)$?

The function $o(G)$

Theorem [M.-S. Lazorec, M. Tărnăuceanu, 2022]

If $G_1 \simeq C_5 \times (C_7 \rtimes C_3)$, then $o(G_1) = \frac{1785}{105} = 17.$

If $G_2 \simeq C_{17} \times (C_7 \rtimes C_3)$, then $o(G_2) = \frac{23205}{357} = 65.$

If $G_3 \simeq C_{85} \times (C_7 \rtimes C_3)$, then $o(G_3) = \frac{487305}{1785} = 273.$

If $G_4 \simeq C_{229} \times C_{13}$, then $o(G_2) = \frac{13446147}{3887} = 285.$

If $G_5 \simeq C_{35} \times (C_{43} \rtimes C_3)$, then $o(G_5) = \frac{1864695}{4515} = 413.$

M.-S. Lazorec, M. Tărnăuceanu, On the average order of a finite group, *to appear*

The function $o(G)$

Problem

What is the structure of a solvable group G such that $o(G) \leq o(\mathcal{A}_5)$?

Remark

$o(G) = o(\mathcal{A}_5)$ if and only if $G \simeq \mathcal{A}_5$

The function $o(G)$

Problem, A.Y. Olshansky

If $c \geq 0$ is any real number, are there only finitely many simple groups G such that $o(G) \leq c$?

Remark

The answer is yes if $c = o(A_5) = 3.5166666\cdots$.
What about $c = 5$?

The function $o(G)$ - another supersolvability criterium

Theorem 14 [M. Tărnăuceanu, 2022]

Let G be a finite group. Suppose that

$$o(G) < \frac{31}{12}$$

Then G is supersolvable.

Moreover $o(G) = \frac{31}{12}$ if and only if $G \simeq \mathcal{A}_4$.

M. Tărnăuceanu, Another criterion for supersolvability of finite groups, *J. Algebra* **604** (2022), 682-693.

Proof of Theorem 13 - some ingredients

Let G be a finite group. Suppose that

$$o(G) \leq o(\mathcal{A}_5).$$

Then either $G \simeq \mathcal{A}_5$ or G is solvable.

Proof.

We use induction on $|G|$.

Write $i_2(G)$ the number of elements of G of order 2.
Write $i_3(G)$ the number of elements of G of order 3.

Then $\psi(G) = 1 \cdot 1 + 2 \cdot i_2(G) + 3 \cdot i_3(G) + \cdots$

Proof of Theorem 13 - some ingredients

Proposition 1 [T.C. Burness, S.D. Scott, 2009]

Let G be a finite group.
If $i_2(G) \geq \frac{3}{4}|G|$, then G is an elementary abelian 2-group.

Let G be a finite non-solvable group.
Then $i_2(G) \leq \frac{4}{15}|G| - 1$.

Let G be a finite non-solvable group.
Then $i_3(G) \leq \frac{7}{20}|G| - 1$.

T.C. Burness, S.D. Scott, On the number of prime order subgroups of finite groups, *J. Australian Math. Soc.* **87** (2009), 329-357.

Proof of Theorem 13 - some ingredients

Let G be a finite group. Write

$$T(G) := \sum_{\chi \in Irr(G)} \chi(1).$$

Then $i_2(G) + 1 \leq T(G)$.

Proof of Theorem 13 - some ingredients

Lemma 1

Let p be a prime and let G be a finite non-solvable group.

If $p \geq 17$ and $o(G) \leq o(\mathcal{A}_5)$,

then G is p-solvable.

Proof of Theorem 13 - some ingredients

Proposition 2 [W.M. Potter, 1988]

Let G be a finite group and let φ an automorphism of G of order 2.

If φ inverts more than $\frac{4}{15}$ elements of G, then G is solvable.

If φ inverts more than $\frac{3}{4}$ elements of G, then G is abelian.

W.M. Potter, Nonsolvable groups with an automorphism inverting many elements, *Arch. Math.* **50** (1988), 292-299.

Proofs - some ingredients

Lemma 2

Let G be a non-solvable finite group and let $\varphi \in Aut(G)$, of order 2.

If φ inverts more than $\frac{2}{9}$ elements of G,

then either G contains a non-trivial normal soluble subgroup,

or $G \simeq \mathcal{A}_5$.

Proof of Theorem 13

Let G be a finite group. Suppose that

$$o(G) \leq o(\mathcal{A}_5).$$

Then either $G \simeq \mathcal{A}_5$ or G is solvable.

Proof.

Suppose that G is a non-solvable finite group with $o(G) \leq o(\mathcal{A}_5)$, $G \not\simeq \mathcal{A}_5$, of minimal order.

First assume that G is simple.

Then G is not p-solvable for every prime p dividing $|G|$.

By Lemma 1,

$$\Pi(G) \subseteq \{2, 3, 5, 7, 11, 13\}.$$

Finite simple groups with $|\Pi(G)| \leq 6$ are known.

A direct check shows that if $o(G) \leq o(\mathcal{A}_5)$, then $G \simeq \mathcal{A}_5$.

Proof of Theorem 13

Now suppose that G is not simple.

Then G has a non-trivial proper normal subgroup M.

$$o(G/M) < o(G) \leq o(\mathcal{A}_5).$$

Thus, by minimality of $|G|$, G/M is solvable.

Then there exists a normal subgroup N of G such that $|G/N| = p$, p a prime.

From $o(G) \leq o(\mathcal{A}_5)$, we get easily that either $p = 2$ or $p = 3$.

Assume that $|G/N| = 2$.

Proof of Theorem 13

We can assume $G = N\langle x\rangle$**,** $o(x) = 2$

Then $\psi(G) = \psi(N) + \psi(xN)$**.**

Write $X := \{xn \mid n \in N, o(xn) = 2\}$.

Obviously $o(xn) = 2$ if and only if $n^x = n^{-1}$.

Thus Lemma 2 applies:

$|X| \leq \frac{2}{9}|N|$**, or** $N \simeq \mathcal{A}_5$**,**

or N **has a non-trivial normal solvable subgroup.**

If $N \simeq \mathcal{A}_5$, then it is easy to see that $G \simeq \mathcal{S}_5$, thus $o(G) = \frac{501}{120} > o(\mathcal{A}_5)$, a contradiction.

If N has a non-trivial normal solvable subgroup, then G has a non-trivial normal solvable subgroup S.

Proof of Theorem 13

From $o(G/S) < o(\mathcal{A}_5)$, we obtain G/S solvable and then G is solvable, a contradiction.

Finally, suppose $|X| \leq \frac{2}{9}|N|$.

Then $|xN \setminus X| \geq |xN| - \frac{2}{9}|N| = \frac{7}{9}|N|$, and we have

$\psi(G) = \psi(N) + \psi(xN) \geq \psi(N) + 2|N| + 2(|xN \setminus X|)$, and

$$\psi(G) \geq \psi(N) + 2|N| + 2\frac{7}{9}|N|.$$

Hence $o(G) \geq \frac{1}{2}o(N) + 1 + \frac{7}{9} = \frac{1}{2}o(N) + 1.777$.

Thus $o(N) < 2(o(G) - 1.777) = 3.486 < o(\mathcal{A}_5)$.

By the minimality of G, we have N solvable, then G is solvable, a contradiction.

Two other functions on the element orders

Let G be a finite group.

Definitions

$$\psi(G)'' := \frac{1}{|G|^2} \sum_{x \in G} o(x).$$

$$\rho(G) := \prod_{x \in G} o(x).$$

The function $\psi(G)''$

Theorem [M. Tărnăuceanu, 2020]

If $\psi''(G) \geq \frac{27}{64}$, then G is cyclic.

If $\psi''(G) \geq \frac{7}{16}$, then G is abelian.

If $\psi''(G) \geq \frac{13}{36}$, then G is nilpotent.

If $\psi''(G) \geq \frac{31}{144}$, then G is supersolvable.

If $\psi''(G) \geq \frac{211}{3600}$, then G is solvable.

M. Tărnăuceanu, Detecting structural properties of finite groups by the sum of element orders, *Israel J. Math.* **238** (2020), 629-637.

The function $\rho(G)$

Theorem [M. Garonzi, M. Patassini, 2016]

Let G be a finite group, $|G| = n$. Then

$$\rho(G) \leq \rho(C_n).$$

Moreover

$$\rho(G) = \rho(C_n) \text{ if and only if } G \simeq C_n.$$

M. Garonzi, M. Patassini, Inequalities detecting structural properties of a finite group, *Comm. Algebra* **45** (2016), 677-687.

The function $\rho(G)$

Theorem [E. Di Domenico, C. Monetta, M. Noce, 2022]

Let G be a finite non-cyclic group with a Sylow tower, $|G| = n$. Then

$$\rho(G) \leq q^{-q} \rho(C_n),$$

where q is the smallest prime dividing n.

E. Di Domenico, C. Monetta, M. Noce, Upper bounds for the product of element orders of finite groups, *to appear*

The function $\rho(G)$

Problem

Is the result true for any finite group?

Theorem [E. Di Domenico, C. Monetta, M. Noce, 2022]

Let G be a finite nilpotent non-cyclic group, $|G| = n$. Then

$$\rho(G) \leq q^{-\frac{n}{q}(q-1)} \rho(C_n),$$

where q is the smallest prime dividing n.

E. Di Domenico, C. Monetta, M. Noce, Upper bounds for the product of element orders of finite groups, *to appear*

The function $\rho(G)$

Problem

Is this bound true for any finite group?

Lemma

Let $G = A \times B$, with $|A|$ and $|B|$ coprime.

Then $\rho(G) = \rho(A)^{|B|}\rho(B)^{|A|}$.

E. Di Domenico, C. Monetta, M. Noce, Upper bounds for the product of element orders of finite groups, *to appear*

Thank you for the attention !

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