Revisiting the sequence reconstruction problem

Elena V. Konstantinova
China Three Gorges Mathematical Research Center, CTGU, Yichang, China
Sobolev Institute of Mathematics & Novosibirsk State University, Russia
e_konsta@math.nsc.ru

This is joint work with Xiang Wang.

In this talk we consider the sequence reconstruction problem which was originally stated in coding theory but it has a combinatorial context. The problem was proposed by V. Levenshtein [1] in 2001 as a local reconstruction of sequences in the model where the same sequence is transmitted over multiple channels, and the decoder receives all the distinct outputs. This problem is stated as follows.

Let S be a set of all sequences of length n, ρ be a metric in S, and $B_r(x) = \{y \in S | \rho(x, y) \leqslant r\}$ be a metric ball of radius r centered at $x \in S$. For any integer $d \geqslant 1$, the minimum number of transmission channels has to be greater than the maximum intersection of two metric balls centered at elements of S denoted as follows:

$$N(n, d, r) = \max_{x_1, x_2 \in S, \rho(x_1, x_2) \geqslant d} |B_r(x_1) \cap B_r(x_2)|. \tag{1}$$

The problem of determining (1) is the *sequence reconstruction problem.*

In this talk we are interested on this problem with respect to permutations under different metrics. First, we give a short overview of some old results. Second, we present some new results for permutations over the Hamming metric [2]. We define the Cayley graph $\mathrm{Sym}_n(H), n \geqslant 2$, over the symmetric group Sym_n generated by cycles of length at least two and show that the distance between two permutations in this graph is the Hamming distance. It is shown that this graph is not distance regular but since it is a vertex-transitive we study the following value:

$$N(n, d, r) = \max_{\pi \in \mathrm{Sym}_n \pi \neq I_n} |B_r(I_n) \cap B_r(\pi)|, \tag{2}$$

where I_n is the identity permutation, the distance between π and I_n is at least d, and $\frac{d}{2} \leqslant r \leqslant n$.

The following two results on (2) are discussed.

Theorem 1. *For any $n \geqslant 2r$ and $t \geqslant 2$, we have:*

$$N(n, 2r, r) = \begin{cases} \binom{2t}{t}, & \text{if } r = 2t; \\ 2\binom{2t-2}{t-1}, & \text{if } r = 2t + 1. \end{cases}$$

Theorem 2. *For any $n \geqslant 2r - 1$ and $t \geqslant 2$, we have:*

$$N(n, 2r - 1, r) \geqslant 2 \cdot \binom{2t-3}{t} + \ell^* \cdot \binom{2t-2}{t-1},$$

where $\ell^* = 5$ if $r = 2t$ and $\ell^* = 7$ if $r = 2t + 1$.

There is an open question whether this bound is attained for any $t \geqslant 5$.

Acknowledgement. This work was supported by the Mathematical Center in Akademgorodok, under agreement No. 075-15-2022-281 with the Ministry of Science and High Education of the Russian Federation.

References

[1] V. I. Levenshtein. Efficient reconstruction of sequences. *IEEE Trans. on Inform. Theory*, **47**: 1 (2001), 2–22.

[2] X. Wang and E. Konstantinova. The sequence reconstruction problem for permutations with the Hamming distance. `https://doi.org/10.48550/arXiv.2210.11864`.